Fridrixs modellari tenzor yig'indisining spektri haqida

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Annotatsiya: Ushbu maqolada panjaradagi uchta zarrachalar sistemasiga mos model operator Gilbert fazosidagi chiziqli chegaralangan va o'z-o'ziga qo'shma operator sifatida qaralgan. Bu operator bir o'lchamli qo'zg'alishga ega Fridrixs modelining tenzor yig'indisi sifatida tasvirlangan. Fridrixs modeli muhim spektrdan chapda yotuvchi yagona oddiy xos qiymatga ega ekanligi ko'rsatilgan. Operatorlar tenzor yig'indisining spektri haqidagi teoremadan foydalanib dastlabki operatorning muhim va diskret spektrlari tadqiq qilingan. Bu model operator ham yagona soda xos qiymatga ega ekanligi isbotlangan.

Kalit so'zlar: model operator, Fridrixs modeli, tenzor yig'indi, muhim spektr, diskret spektr, xos qiymat.

About the spectrum of the tensor set of Friedrichs models

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Abstract: In this paper, a model operator corresponding to a three-particle system in a grid is considered to be a linear boundary in Gilbert space and a self-contained operator. This operator is described as a tensor sum of the Friedrichs model with one-dimensional motion. The Friedrichs model is shown to have a single simple characteristic that lies to the left of the important spectrum. Important and discrete spectra of the original operator were studied using the theorem on the spectrum of the sum of tensors of operators. This model operator has also been proven to have a unique soda specific value.

Keywords: model operator, Friedrichs model, tensor sum, significant spectrum, discrete spectrum, eigenvalue.

Zamonaviy matematik fizika masalalarida, xususan, statistik fizika, qattiq jismlar fizikasi, kvant maydon nazariyasi va yana ko'plab sohalarda panjaradagi uchta zarrachalar sistemasiga mos model operatorlarning spektral xossalarini o‘rganish bilan bog‘liq masalalar uchrab turadi. Mazkur maqola shu turdagi model operatorlardan birining muhim va diskret spektrini o‘rganish masalasiga bag‘ishlangan.

$L_2^s[-\pi;\pi]^2$ orqali $[-\pi;\pi]^2$ to‘plamda aniqlangan kvadrati bilan integrallanuvchi (umuman olganda kompleks qiymatlarni qabul qiluvchi) simmetrik funksiyalarning Gilbert fazosini belgilaymiz. Bu fazodan olingan f va g elementlar uchun

$$(f,g)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x,y)\overline{g(x,y)}\,dxdy;$$

$$\|f\|=\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^2\,dxdy\right)^{\frac{1}{2}}$$

tengliklar o‘rinlidir.

Ushbu maqolada $L_2^s[-\pi;\pi]^2$ Gilbert fazosida

$$H=H_0-V_1-V_2 \qquad (1)$$

kabi aniqlanuvchi H model operatorni qaraymiz. Bunda H_0 ko‘paytirish operatori:

$$(H_0f)(x,y)=(2-cosx-cosy)f(x,y),$$

V_1 va V_2 operatorlar esa quyidagicha ta’sir qiluvchi xususiy integralli operatorlar:

$$(V_1f)(x,y)=siny\int_{-\pi}^{\pi}sint\,f(x,t)dt;$$

$$(V_2f)(x,y)=sinx\int_{-\pi}^{\pi}sint\,f(t,y)dt.$$

Dastlab H operatorni chiziqlilikga tekshiramiz. Buning uchun ixtiyoriy $\alpha,\beta\in C$ kompleks sonlari va ixtiyoriy $f,g\in L_2^s[-\pi;\pi]^2$ elementlar uchun

$$H(\alpha f+\beta g)=\alpha Hf+\beta Hg$$

tenglik bajarilishini ko‘rsatamiz.

Haqiqatan ham,

$$H(\alpha f+\beta g)(x,y)=(2-cosx-cosy)\big(\alpha f(x,y)+\beta g(x,y)\big)$$

$$-siny\int_{-\pi}^{\pi}sint\big(\,\alpha f(x,t)+\beta g(x,t)\big)dt-$$

$$-sinx\int_{-\pi}^{\pi}sint\big(\,\alpha f(t,y)+\beta g(t,y)\big)dt=$$

$$\alpha\Bigg[(2-cosx-cosy)f(x,y)-siny\int_{-\pi}^{\pi}sintf(x,t)dt-$$

$$-sinx\int_{-\pi}^{\pi}sint\,f(t,y)dt\Bigg]+$$

$$+\beta\Bigg[(2-cosx-cosy)g(x,y)-siny\int_{-\pi}^{\pi}sint\,g(x,t)dt-$$

$$-sinx\int_{-\pi}^{\pi}sint\;g(t,y)dt\Bigg]=\;\alpha(Hf)(x,y)+\beta(Hg)(x,y).$$

Demak, H chiziqli operator ekan.

H operatorning chegaralanganligini ko‘rsatish uchun $\|Hf\|^2$ ni qaraymiz va uni quyidagicha baholaymiz:

$$\|Hf\|^2=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|(Hf)(x,y)|^2\,dxdy=$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg|(2-cosx-cosy)f(x,y)-siny\int_{-\pi}^{\pi}sintf(x,t)dt-$$

$$-sinx\int_{-\pi}^{\pi}sint\;f(t,y)dt\Bigg|^2\,dxdy\le 3\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|2-cosx-cosy|^2\cdot$$

$$\cdot|f(x,y)|^2dxdy+3\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg|siny\int_{-\pi}^{\pi}sintf(x,t)dt\Bigg|^2\,dxdy+$$

$$+3\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg|sinx\int_{-\pi}^{\pi}sint\;f(t,y)dt\Bigg|^2\,dxdy\le 48\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^2\,dxdy+$$

$$+3\int_{-\pi}^{\pi}sin^2ydy\int_{-\pi}^{\pi}sin^2tdt\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,t)|^2dxdt+$$

$$+3\int_{-\pi}^{\pi}sin^2xdx\int_{-\pi}^{\pi}sin^2tdt\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(t,y)|^2dtdy=$$

$$=(48+6\pi^2)\|f\|^2,$$

ya’ni

$$\|Hf\|\le\sqrt{48+6\pi^2}\,\|f\|.$$

Demak, H chegaralangan operator ekan.

Endi H operatorni o‘z-o‘ziga qo‘shmalikga tekshiramiz:

$$(Hf,g)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(Hf)(x,y)\overline{g(x,y)}\,dxdy=$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg[(2-cosx-cosy)f(x,y)-siny\int_{-\pi}^{\pi}sintf(x,t)dt-$$

$$-sinx\int_{-\pi}^{\pi}sint\;f(t,y)dt\Bigg]\overline{g(x,y)}dxdy=$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(2-cosx-cosy)f(x,y)\overline{g(x,y)}dxdy-$$

$$-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}siny\;sint\;f(x,t)\;\overline{g(x,y)}dxdydt-$$

$$-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}sinx\;sint\;f(t,y)\;\overline{g(x,y)}dxdydt=$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\left[\overline{(2-cosx-cosy)g(x,y)-} \right.$$

$$\left. \overline{-siny\int_{-\pi}^{\pi} sint\ g(x,t)dt - sinx\int_{-\pi}^{\pi} sint\ g(t,y)dt}\right] dxdy =$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\overline{(Hg)(x,y)}\,dxdy = (f,Hg).$$

Shunday qilib, $H^* = H$, ya’ni H o‘z-o‘ziga qo‘shma operator ekan.

(1) formula yordamida ta’sir qiluvchi H operator panjaradagi uchta zarrachalar sistemasiga mos keluvchi model operator hisoblanadi. Bunda H_0 ko‘paytirish operatoriga qo‘zg‘almas operator deyiladi va V_1 hamda V_2 operatorlarga esa lokal bo‘lmagan potensial operatorlari deyiladi.

H operatorning spektral xossalarini o‘rganish maqsadida $L_2[-\pi;\pi]$ Gilbert fazosida

$$h = h_0 - v \qquad (2)$$

ko‘rinishda ta’sir qiluvchi operatorni qaraymiz. Bunda h_0 ko‘paytirish operatori:

$$(h_0 g)(x) = (1-cosx)g(x),$$

v esa integral operator:

$$(vg)(x) = sinx\int_{-\pi}^{\pi} sint\ g(t)dt.$$

Yuqorida keltirilgan mulohazalarga o‘xshash tekshirishlar yordamida (2) formula yordamida ta’sir qiluvchi h operatorning chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma ekanligini ko‘rsatish mumkin.

h operator zamonaviy matematik fizikada Fridrixs modeli nomi bilan mashhur bo‘lib, panjaradagi ikki zarrachali sistemaga mos model operator sifatida ham qarash mumkin.

Quyida Fridrixs modeli haqida dastlabki tushunchalarni keltirib o‘tamiz.

$L_2[-1;1]$ Gilbert fazosida

$$H_\lambda f(x) = xf(x) + \lambda\int_{-1}^{1} K(x;y)f(y)dy$$

ko‘rinishida ta’sir qiluvchi H_λ operatorni qaraymiz. Bu yerda λ haqiqiy musbat son, K(x;y) esa $[-1;1]^2$ da aniqlangan haqiqiy qiymatli uzluksiz simmetrik funksiya, ya’ni K(x;y)=K(y;x) tenglik o‘rinlidir. Bu operator birinchi marta K.Fridrixs tomonidan uzluksiz spektr qo‘zg‘alishlari nazariyasi modeli sifatida [1] ishda qaralgan. Bu maqolada K(x;y) yadro o‘zining o‘zgaruvchilarining uzluksiz funksiyasi bo’lib, Gyolder shartini va

K(x;-1)=K(x;1)=K(-1;y)=K(1;y)=0, x,y∈[-1;1]

shartlarni qanoatlantirishi talab qilingan.

Fridrixs tomonidan $\lambda \in R$ parametrning yetarlicha kichik qiymatlarida H_λ va H_0 operatorlar unitar ekvivalent ekanligi isbotlangan, ya'ni H_λ operator $[-1;1]$ ga teng bo'lgan sodda Lebeg spektriga ega ekanligi ko'rsatilgan. 1948-yilda Fridrixs o'zining [2] ishida o'z modelini quyidagicha umumlashtirish masalasini taklif qilgan: birinchidan, $[-1;1]$ o'rniga haqiqiy sonlar o'qidagi ixtiyoriy chekli yoki cheksiz bo'lgan ℓ intervalni qarash; ikkinchidan, qiymatlari biror abstrakt Gilbert fazosi bo'lgan f funksiyalarni qarash. ℓ interval cheksiz bo'lgan holda K(x;y) yadro cheksizlikda kamayuvchi bo'lsin degan qo'shimcha shart kiritib Fridrixs bu nisbatan umumiy holda H_λ va H_0 operatorlarning unitar ekvivalent ekanligini isbotlagan.

Keyinchalik Fridrixsning [1] va [2] ishlari O.A. Ladijenskiy, .D.Faddeyevlar tomonidan [3] maqolada va L.D.Faddeyev tomonidan [4] maqolada rivojlantirilgan.

Panjaradagi Fridrixs modeli bilan bog'liq tadqiqotlar [5-19] ishlarda olib borilgan. Umumlashgan Fridrixs modelining ayrim spektral xossalari [20-30] ishlarda o'rganilgan. Bu xossalar o'z navbatida panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasiga mos 3-tartibli operatorli matrisalarning muhim va diskret spektrlarini tadqiq qilishda foydalanilgan.

Aniqlanishiga ko'ra v bir o'lchamli operator, ya'ni $dim\, Im\, v = 1$. Haqiqatan ham,

$$Im\, v = \{g(x) = asinx : a \in C\}$$

tenglikdan $dim\, Im\, v = 1$ ekanligi kelib chiqadi. Chekli o'lchamli qo'zg'alishlarda muhim spektrning o'zgarmasligi haqidagi Veyl teoremasiga teoremasiga ko'ra h_0 va $h_0 - v$ operatorlarning muhim spektrlari ustma-ust tushadi. h_0 operator $1 - cosx$ funksiyaga ko'paytirish operatori. Shu bois

$$\sigma_{ess}(h_0) = [0; 2].$$

Yuqorida keltirilgan mulohazalarga ko'ra

$$\sigma_{ess}(h) = [0; 2].$$

h operatorning xos qiymatlarini o'rganish maqsadida $C\backslash[0; 2]$ sohada regulyar bo'lgan

$$\Delta(z) = 1 - \int_{-\pi}^{\pi} \frac{sin^2 t dt}{1 - cost - z}$$

funksiyani qaraymiz. Zamonaviy matematik fizikada $\Delta(\cdot)$ funksiyaga h operatorga mos Fredgolm determinanti deyiladi.

Quyidagi lemma h operator xos qiymatlari va $\Delta(\cdot)$ funksiya nollari orasidagi bog'lanishni ifodalaydi.

1-lemma. $z \in C\backslash[0; 2]$ soni h Fridrixs modelining xos qiymati bo'lishi uchun $\Delta(z) = 0$ bo'lishi zarur va yetarlidir.

Isboti. Zaruriyligi. Faraz qilaylik, $z \in C\backslash[0;2]$ soni h Fridrixs modelining xos qiymati, $g \in L_2[-\pi;\pi]$ esa unga mos xos funksiya bo‘lsin. U holda $g(\cdot)$ xos funksiya $hg = zg$ xos qiymatga nisbatan tenglamani qanoatlantiradi.

$hg = zg$ tenglamani

$$(1 - cosx - z)g(x) - sinx \int_{-\pi}^{\pi} sint\ g(t)dt = 0 \quad (3)$$

ko‘rinishida yozib olamiz. Ushbu

$$a = \int_{-\pi}^{\pi} sint\ g(t)dt \quad (4)$$

belgilashni kiritamiz. $z \in C\backslash[0;2]$ ekanligidan barcha $x \in [-\pi;\pi]$ lar uchun $1 - cosx - z \neq 0$ munosabat kelib chiqadi. Shu bois (3) tenglamadan $g(x)$ xos funksiya uchun

$$g(x) = \frac{asinx}{1 - cosx - z} \quad (5)$$

ifodaga ega bo‘lamiz. $g(x)$ xos funksiya uchun topilgan (5) ifodani (4) belgilashga qo‘yamiz:

$$a = \int_{-\pi}^{\pi} sint\ \frac{asint}{1 - cost - z} dt$$

yoki

$$a\left(1 - \int_{-\pi}^{\pi} \frac{sin^2 t\ dt}{1 - cost - z}\right) = 0$$

yoki

$$a \cdot \Delta(z) = 0.$$

Agar oxirgi tenglikda $a = 0$ bo‘lsa, u holda (5) tenglikga ko‘ra $g(x) = 0$ bo‘ladi. Bu esa $g(x)$ funksiyaning xos funksiya ekanligiga, ya’ni nolmas funksiya ekanligiga zid. Shu sababli $\Delta(z) = 0$ bo‘ladi. 1-lemmaning zaruriylik qismi bajarildi.

Yetarliligi. Faraz qilaylik, biror $z_0 \in C\backslash[0;2]$ soni uchun $\Delta(z_0) = 0$ bo‘lsin. U holda

$$g(x) = \frac{sinx}{1 - cosx - z_0}$$

funksiya $hg = z_0 g$ tenglamani qanoatlantiradi:

$$(hg)(x) - z_0 g(x) = (1 - cosx - z_0)g(x) - sinx \int_{-\pi}^{\pi} sint\ g(t)dt =$$

$$(1 - cosx - z_0)\frac{sinx}{1 - cosx - z_0} - sinx \int_{-\pi}^{\pi} \frac{sin^2 t\ dt}{1 - cost - z_0} =$$

$$sinx\left(1 - \int_{-\pi}^{\pi} \frac{sin^2 t\ dt}{1 - cost - z_0}\right) = sinx \cdot \Delta(z_0) = 0.$$

$g(x)$ funksiya $[-\pi;\pi]$ da aniqlangan uzluksiz funksiya. Shu sababli $g \in L_2[-\pi;\pi]$ munosabat o‘rinli bo‘ladi. 1-lemma to‘liq isbotlandi.

Isbot qilingan 1-lemmadan h operatorning diskret spektri uchun quyidagi natija kelib chiqadi.

1-natija. h Fridrixs modelining diskret spektri uchun

$$\sigma_{disc}(h) = \{z \in C\backslash[0;2]: \Delta(z) = 0\}$$

tenglik o‘rinlidir.

h Fridrixs modelining xos qiymatlarini o‘rganishda

$$I(z) = \int_{-\pi}^{\pi} \frac{sin^2t\,dt}{1 - cost - z}$$

integralning $z = 0$ nuqtadagi qiymati muhim ahamiyat kasb etadi.

$z = 0$ nuqtadagi qiymatini hisoblaymiz:

$$I(0) = \int_{-\pi}^{\pi} \frac{sin^2t\,dt}{1 - cost} = \int_{-\pi}^{\pi} \frac{1 - cos^2t}{1 - cost} dt =$$

$$= \int_{-\pi}^{\pi} \frac{(1 - cost)(1 + cost)}{1 - cost} dt = \int_{-\pi}^{\pi} (1 + cost)dt = 2\pi.$$

Integral belgisi ostida limitga o‘tish haqidagi Lebeg teoremasiga ko‘ra

$$\Delta(0) = \lim_{z\to-0} \Delta(z) = \lim_{z\to-0}\big(1 - I(z)\big) = 1 - I(0) = 1 - 2\pi < 0.$$

$\Delta(\cdot)$ funksiya $(-\infty; 0)$ oraliqda uzluksiz, monoton kamayuvchi va $\Delta(0) = 1 - 2\pi < 0$ hamda

$$\lim_{z\to-\infty} \Delta(z) = 1$$

ekanligini inobatga olsak, u holda shunday $z_0 \in (-\infty; 0)$ soni topilib, $\Delta(z_0) = 0$ bo‘ladi. Bunda z_0 soni $\Delta(\cdot)$ funksiyaning oddiy (bir karrali) noli bo‘ladi. 1- lemmaga ko‘ra $z_0 \in (-\infty; 0)$ soni h Fridrixs modelining oddiy (bir karrali) xos qiymati bo‘ladi.

v operatorning musbat aniqlangan ekanligini ekanligini ko‘rsatamiz: ixtiyoriy $f \in L_2[-\pi;\pi]$ uchun

$$(vf, f) = \int_{-\pi}^{\pi} (vf)(x)\overline{f(x)}dx = \int_{-\pi}^{\pi}\left(sinx \int_{-\pi}^{\pi} sint\, f(t)dt\right)\overline{f(x)}dx =$$

$$= \int_{-\pi}^{\pi} sinx\, f(x)dx \cdot \overline{\int_{-\pi}^{\pi} sint\, f(t)dt} = \left|\int_{-\pi}^{\pi} sinx\, f(x)dx\right|^2 \geq 0$$

tengsizlik o‘rinli. Shu sababli v musbat aniqlangan operator.

Faraz qilaylik, $z > 2$ bo‘lsin. U holda ixtiyoriy $f \in L_2[-\pi;\pi]$ uchun

$$(hf, f) - z(f, f) = \big((h_0 - Iz)f, f\big) - (vf, f) =$$

$$= \int_{-\pi}^{\pi} (1 - cosx - z)|f(x)|^2 dx - (vf, f) < 0$$

tengsizlik o‘rinli. Bu esa o‘z navbatida h Fridrixs modeli muhim spektrdan o‘ngda joylashgan xos qiymatlarga ega emasligini bildiradi. Shunday qilib, h Fridrixs modelining spektri uchun

$$\sigma(h) = \{z_0\} \cup [0; 2], z_0 < 0$$

tenglik o‘rinlidir. Bu yerda z_0 soni $\Delta(\cdot)$ funksiyaning yagona manfiy noli.

H va h operatorlarning aniqlanishiga ko‘ra

$$H = h\otimes I + I\otimes \qquad (6)$$

tasvir o‘rinlidir. Operatorlar tenzor yig‘indisining spektri haqida teoremaga ko‘ra

$$\sigma(H) = \sigma(h) + \sigma(h)$$

tenglik o‘rinlidir. Bu yerda $A, B \subset R$ soni to‘plamlar uchun $A + B$ to‘plam

$$A + B = \{a + b : a \in A, b \in B\}$$

kabi aniqlanadi.

Shu sababli

$$\sigma(H) = \sigma(h) + \sigma(h) = \{\{z_0\} \cup [0;2]\} + \{\{z_0\} \cup [0;2]\} =$$
$$\{2z_0\} \cup [z_0;\ z_0 + 2] \cup [0;4]$$

tenglik o‘rinlidir.

H operatorning muhim va diskret spektrlari, haqidagi asosiy teorema yuqorida keltirilgan mulohazalardan xulosa sifatida kelib chiqadi:

Asosiy teorema: H operatorning muhim spektri uchun

$$\sigma_{ess}(H) = [z_0; z_0 + 2] \cup [0;4]$$

tenglik o‘rinli, bu yerda z_0 soni $\Delta(\cdot)$ funksiyaning yagona manfiy noli. Bundan tashqari, $2z_0$ soni H operatorning yagona oddiy (bir karrali) xos qiymati bo‘ladi, ya’ni

$$\sigma_{disc}(H) = \{2z_0\}.$$

Xulosa o‘rnida shuni aytish mumkinki, maqolada tadqiq qilingan H operatorni (6) ko‘rinishida tasvirlash mumkinligi H operator muhim va diskret spektrlarini aniq hisoblash imkonini bermoqda. Muhim spektr oralig‘ida bo‘shliq hosil bo‘lishi z_0 xos qiymatning joylashgan oralig‘idan (aniq qiymatidan) bog‘liq. Uni Maple amaliy maketi yordamida hisoblash mumkin.

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