Diskret parametrli ikkinchi tartibli operatorli matritsa xos qiymatlarining mavjudligi

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Annotatsiya: Mazkur maqolada Fok fazosining ikki zarrachali qirqilgan qism fazosida aniqlangan diskret parametrli ikkinchi tartibli operatorli matritsa tadqiq qilingan. Unga mos Fredgolm determinanti muhim spektrdan chapda va o‘ngda joylashgan oraliqlarda (haqiqiy sonlar o‘qining qism to‘plamlarida) monoton kamayuvchi funksiya ekanligi ko‘rsatilgan. O‘rganilayotgan operatorli matritsa xos qiymatlarining mavjudlik shartlari topilgan.

Kalit so’zlar: operatorli matritsa, diskret parametr, spektr, muhim spektr, diskret spektr, Fredgolm determinanti.

Presence of discrete parameter second-order operator matrix-specific values

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Abstract: This paper examines a matrix with a second-order operator with a discrete parameter defined in the space of a two-particle fragment of the Fock space. The corresponding Fredholm determinant is shown to be a monotonically decreasing function in the intervals to the left and right of the significant spectrum (in the partial sets of the axes of real numbers). The conditions for the existence of matrix-specific values of the operator under study are found.

Keywords: operator matrix, discrete parameter, spectrum, critical spectrum, discrete spectrum, Fredholm determinant.

Kvant mexanikasi, statistik mexanika va gidrodinamikaning ko‘plab masalalalari Fridrixs tomonidan uzluksiz spektr qo‘zg‘alishlari nazariyasi modeli sifatida kiritilgan Fridrixs modeli [1-4] deb nomlanuvchi operatorni o‘rganish masalasiga keltiriladi. Panjaradagi Fridrixs modeli bilan bog’liq tadqiqotlar [5-19] ishlarda olib borilgan. Aynan bu turdagi operatorlar yordamida uch zarrachali Shryodinger operatori va unga mos model operatorlarning muhim va diskret spektrlari o‘rganiladi Umumlashgan

Fridrixs modelining ayrim spektral xossalari [20-30] ishlarda o'rganilgan. Bu xossalar o'z navbatida panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasiga mos 3-tartibli operatorli matrisalarning muhim va diskret spektrlarini tadqiq qilishda foydalanilgan.

H_0:=C orqali bir o'lchamli kompleks fazoni, H_1:=$L_2[-\pi;\pi]$ orqali, $[-\pi;\pi]$ da aniqlangan kvadrati bilan integrallanuvchi (umuman olganda kompleks qiymatlarni qabul qiluvchi) funksiyalarning Gilbert fazoni belgilaymiz. H_1 va H_2 fazolarning to'g'ri yig'indisini H orqali belgilaymiz, ya'ni $H:=H_0\oplus H_1$. Odatda H.Gilbert fazosiga Fok fazosining qirqilgan ikki zarrachali qism fazosi deyiladi.

Ushbu maqolada H.Gilbert fazosida quyidagi ko'rinishdagi diskret parametrli ikkinchi tartibli operatorli matritsalarning spektral xossalari o'rganiladi:

$$A_\alpha(m) = \begin{pmatrix} A_{00}(m) & \alpha A_{01} \\ \alpha A_{01}^* & A_{11}(m) \end{pmatrix}.$$

Bunda

$$A_{ii}(m): H_i \to H_i,\ i=0,1 \text{ va } A_{01}: H_1 \to H_0$$

matritsaviy elementlar quyidagi tengliklar yordamida ta'sir qiladi:

$$A_{00}(m)f_0=m\varepsilon f_0,\ f_0\in H_0;$$

$$A_{01}f_1=\int_{-\pi}^{\pi} v(t)f_1(t)dt,\ f_1\in H_1;$$

$$(A_{11}(m)f_1)(x)=(m\varepsilon+u(x))f_1(x),\ f_1\in H_1.$$

Bunda $m\in Z$; $\varepsilon > 0$; $\alpha > 0$; v(·) va u(·) funksiyalar $[-\pi;\pi]$ da aniqlangan haqiqiy qiymatli uzluksiz funksiyalar.

A_{01}^* orqali A_{01} operatorga qo'shma operator belgilangan bo'lib, bu operator

$$A_{01}^*:H_0\to H_1,\ (A_{01}^*f_0)(x)=v(x)f_0,\ f_0\in H_0$$

tenglik bilan aniqlanadi.

Haqiqatdan ham

$$(A_{01}f_1,f_0)_0=A_{01}f_1\overline{f_0}=\int_{-\pi}^{\pi} v(t)f_1(t)dt\cdot\overline{f_0}=\int_{-\pi}^{\pi} f_1(t)(v(t)\overline{f_0})dt=$$

$$=\int_{-\pi}^{\pi} f_1(t)\overline{(v(t)f_0)}dt=(f_1,A_{01}^*f_0)_1.$$

Bunda biz v(x) funksiyaning haqiqiy qiymatli funksiya ekanligidan, ya'ni

$$\overline{v(x)}=v(x),\ x\in[-\pi;\pi]$$

ekanligidan foydalandik. Yuqoridagi tenglikka ko'ra

$$(A_{01}^*,f_0)(x)=v(x)f_0,\ f_0\in H_0$$

tenglik o'rinli ekan.

$A_\alpha(m)$ operatorning muhim spektrini o'rganish maqsadida uni

$$A_\alpha(m)=A_0(m)+\alpha V$$

ko'rinishida tasvirlab olamiz.

Bu yerda $A_0(m)=A_\alpha(m)/_{\alpha=0}$, ya'ni

$$A_0(m)=\begin{pmatrix} A_{00}(m) & 0 \\ 0 & A_{11}(m) \end{pmatrix},$$

V-operator esa V=$\frac{1}{\alpha}$ ($A_\alpha(m) - A_0(m)$) kabi aniqlangan, ya'ni

$$V=\begin{pmatrix} 0 & A_{01} \\ A_{01}^* & 0 \end{pmatrix}$$

V operator 2 o'lchamli operator ekanligini oson ko'rsatish mumkin.

Chekli o'lchamli qo'zg'alishlarda muhim spektrning o'zgarmasligi haqidagi mashhur Veyl teoremasiga ko'ra $A_\alpha(m)$ va $A_0(m)$ operatorlarning muhim spektrlari ustma-ust tushadi. Aniqlanishiga ko'ra $A_0(m)$ diagonal operatordir. Shuning uchun

$$\sigma(A_0(m))=\sigma(A_{00}(m)) \cup \sigma(A_{11}(m))$$

tenglik o'rinli bo'ladi. $A_{00}(m)$ operatorning spektri

$$\sigma\big(A_{00}(m)\big) = \sigma_{disc}(A_{00}(m))=\{\mathrm{m}\varepsilon\}$$

kabi aniqlanadi.

A_{11}(m) operatorning spektri esa sof muhim spektr bo'lib,

$$\sigma\big(A_{11}(m)\big) = \sigma_{ess}(A_{11}(m))=[\ \mathrm{m}\varepsilon + u_{min}, \mathrm{m}\varepsilon + u_{max}]$$

tenglik o'rinlidir.

Bu yerda $A_{11}(m)$ operator $\mathrm{m}\varepsilon + u(\cdot)$ funksiyaga ko'paytirish operatori bo'lganligi bois, uning spektri sof muhim spektr bo'lib, $\mathrm{m}\varepsilon + u(\cdot)$ funksiyaning qiymatlar sohasi bilan ustma-ust tushadi.

u_{min} va u_{max} orqali

$$u_{min}=\min_{-\pi\le x\le\pi} u(x),\ u_{max}=\max_{-\pi\le x\le\pi} u(x)$$

sonlari belgilangan.

Demak $A_\alpha(m)$ operatorning muhim spektri α ta'sirlashish parametriga bog'liq bo'lmasdan, u uchun

$$\sigma_{ess}(A_\alpha(m))=[\ \mathrm{m}\varepsilon + u_{min}, \mathrm{m}\varepsilon + u_{max}]$$

Endi $A_\alpha(m)$ operatorning diskret spektrini o'rganinish masalasini qaraymiz.Ushbu maqsadda $\mathbb{C}\backslash[\ \mathrm{m}\varepsilon + u_{min}, \mathrm{m}\varepsilon + u_{max}]$ sohada regulyar bo'lgan

$$\Delta_\alpha(m;z)= \mathrm{m}\varepsilon - z - \alpha^2 \int_{-\pi}^{\pi} \frac{\vartheta^2(t)dt}{u(t)-z}$$

funksiyani qaraymiz.

Odatda $\Delta_\alpha(m;\cdot)$ funksiyaga $A_\alpha(m)$ operatorga mos Fredgolm determinanti deyiladi. Quyidagi lemma $A_\alpha(m)$ operatorning xos qiymatlari va $\Delta_\alpha(m;\cdot)$ funksiya nollari orasidagi bog'lanishni ifodalaydi.

1-lemma. *$Z\in \mathbb{C}\backslash[\ m\varepsilon + U_{min}, m\varepsilon + U_{max}]$ soni $A_\alpha(m)$ operatorning xos qiymati bo'lishi uchun $\Delta_\alpha(m;z) = 0$ bo'lishi zarur va yetarlidir.*

$\Delta_\alpha(m;z)$ Fredgolm determinantining asosiy xossalarini bayon qilamiz va isbotlaymiz.

1-xossa. $\Delta_\alpha(m;\cdot)$ *Fredgolm determinanti (-∞ ; $m\varepsilon + u_{min}$) va ($m\varepsilon + u_{max}$; ∞) oraliqlarda monoton kamayuvchi funksiya bo'ladi.*

Isbot: 1- xossani isbotlash maqsadida $\Delta_\alpha(m;\cdot)$ funksiyaning 1-tartibli hosilasini hisoblaymiz:

$$\frac{d}{dz}\Delta_\alpha(m;\ z) = \frac{d}{dz}\left(\mathrm{m}\varepsilon - z - \alpha^2\int_{-\pi}^{\pi}\frac{\vartheta^2(t)dt}{m\varepsilon+u(t)-z}\right) = -1 - \alpha^2\int_{-\pi}^{\pi}\frac{v^2(t)dt}{(m\varepsilon+u(t)-z)^2}$$

Ko‘rinib turibdiki ixtiyoriy z∈(-∞ ; $m\varepsilon+u_{min}$)∪($m\varepsilon+u_{max}$; ∞) uchun $\frac{d}{dz}\Delta_\alpha(m;\ z) < 0$tengsizlik bajariladi. Kesmada uzluksiz funksiyaning monotonligi haqidagi teoremaga ko‘ra $\Delta_\alpha(m;\ z)$ funksiya (-∞ ; $m\varepsilon+u_{min}$) va ($m\varepsilon+u_{max}$; ∞) oraliqlarda monoton kamayuvchi bo‘ladi. 1-xossa to‘liq isbotlandi.

2-xossa. *$A_\alpha(m)$ operator ko‘pi bilan 2 ta xos qiymatga ega bo‘ladi. Ularning har biri sodda xos qiymatlar bo‘lib, biri $m\varepsilon+u_{min}$ dan chapda , ikkinchisi esa $m\varepsilon+u_{max}$ dan o‘ngda joylashgan bo‘ladi.*

Isbot: $\Delta_\alpha(m;\cdot)$ funksiyaning aniqlanishiga ko‘ra

$$\lim_{z\to-\infty}\Delta_\alpha(m;\ z) = +\infty$$

tenglik o‘rinli bo‘ladi. 1-xossaga ko‘ra $\Delta_\alpha(m;\cdot)$ funksiya (-∞ ; $m\varepsilon+u_{min}$) intervalda monoton kamayuvchi. Shu sababli $\Delta_\alpha(m;\cdot)$ funksiya (-∞ ; $m\varepsilon+u_{min}$) oraliqda ko‘pi bilan bitta nolga ega. 1-lemmaga ko‘ra bu funksiyaning noli $A_\alpha(m)$ operatorning xos qiymati bo‘ladi. Demak $A_\alpha(m)$ operator (-∞ ; $m\varepsilon+u_{min}$) oraliqda ko‘pi bilan bitta xos qiymatga ega bo‘ladi. Xuddi shu kabi

$$\lim_{z\to\infty}\Delta_\alpha(m;\ z) = -\infty$$

tenglik o‘rinlidir. 1-xossaga ko‘ra $\Delta_\alpha(m;\cdot)$ funksiya ($m\varepsilon+u_{max}$; ∞) intervalda monoton kamayuvchi. Shu sababli $\Delta_\alpha(m;\cdot)$ funksiya ($m\varepsilon+u_{max}$; ∞) oraliqda ko‘pi bilan bitta nolga ega. 1-lemmaga ko‘ra bu funksiyaning noli $A_\alpha(m)$ operatorning xos qiymati bo‘ladi. Demak $A_\alpha(m)$ operator ($m\varepsilon+u_{max}$; ∞) oraliqda ko‘pi bilan bitta xos qiymatga ega bo‘ladi. 1 – xossa to‘liq isbotlandi.

3-xossa. *$A_\alpha(m)$ operator $z_0 \leq m\varepsilon+u_{min}$ sonidan kichik xos qiymatga ega bo‘lishi uchun $\Delta_\alpha(m; z_0) < 0$ bo‘lishi zarur va yetarlidir.*

Isbot: Faraz qilaylik $A_\alpha(m)$ operator $z_0 \leq \mathrm{m}\varepsilon+\mathrm{u}_{min}$ sonidan kichik xos qiymatga ega bo‘lsin. U holda 1-lemmaga ko‘ra shunday $z < z_0$ soni topilib, $\Delta_\alpha(m;\ z) = 0$ tenglik bajariladi. $\Delta_\alpha(m;\cdot)$ funksiyaning monoton kamayuvchi ekanligiga ko‘ra

$$\Delta_\alpha(m; z_0) < \Delta_\alpha(m;\ z) = 0,$$

ya’ni

$$\Delta_\alpha(m; z_0) < 0.$$

Yetarliligi: Faraz qilaylik $\Delta_\alpha(m; z_0) < 0$ bo‘lsin. U holda

$$\lim_{z\to-\infty}\Delta_\alpha(m;\ z) = +\infty$$

munosabatga va uzluksiz funksiyaning xossasiga hamda $\Delta_\alpha(m;\cdot)$ funksiyaning monotonligiga ko‘ra yagona $z\epsilon$(-∞; z_0) soni topilib, $\Delta_\alpha(m;\ z) = 0$ bo‘ladi.

1-lemmaga ko‘ra bu z soni $A_\alpha(m)$ operatorning z_0 sonidan chapda yotuvchi xos qiymati bo‘ladi. 3-xossa to‘liq isbotlandi.

4-xossa: $A_\alpha(m)$ *operator* $z_1 \geq m\varepsilon + u_{max}$ *sonidan katta xos qiymatga ega bo'lishi uchun* $\Delta_\alpha(m; z_1) > 0$ *bo'lishi zarur va yetarlidir.*

Isbot: Faraz qilaylik $A_\alpha(m)$ operator $z_1 \geq m\varepsilon + u_{max}$ sonidan katta xos qiymatga ega bo‘lsin. U holda 1-lemmaga ko‘ra shunday $z > z_1$ soni topilib, $\Delta_\alpha(m; z) = 0$ tenglik bajariladi. $\Delta_\alpha(m; \cdot)$ funksiyaning monoton kamayuvchi ekanligiga ko‘ra

$$\Delta_\alpha(m; z_1) > \Delta_\alpha(m; z) = 0,$$

ya’ni

$$\Delta_\alpha(m; z_1) > 0.$$

Yetarliligi: Faraz qilaylik $\Delta_\alpha(m; z_1) > 0$ bo‘lsin. U holda $\lim_{z\to\infty} \Delta_\alpha(m; z) = -\infty$ munosabatga va uzluksiz funksiyaning xossasiga hamda $\Delta_\alpha(m; \cdot)$ funksiyaning monotonligiga ko‘ra yagona $z\epsilon(z_1;\infty)$ soni topilib , $\Delta_\alpha(m; z) = 0$ bo‘ladi.2.1.1-lemmaga ko‘ra bu z soni $A_\alpha(m)$ operatorning z_1 dan o‘ngda yotuvchi xos qiymati bo‘ladi. 4-xossa to‘liq isbotlanadi.

$\Delta_\alpha(m; \cdot)$ funksiyaning $(-\infty\,; m\varepsilon + u_{min})$ oraliqda monoton kamayuvchi ekanligidan chekli yoki cheksiz bo‘lgan

$$\lim_{z\to m\varepsilon+u_{min}-0} \Delta_\alpha(m; z) = \Delta_\alpha(m; m\varepsilon + u_{min})$$

limitga ega bo‘lamiz. Bu yerda biz integral belgisi ostida limitga o‘tish mumkinligi haqidagi Lebeg teoremasidan foydalandik va

$$\Delta_\alpha(m; m\varepsilon + u_{min}) = m\varepsilon - m\varepsilon - u_{min} - \alpha^2 \int_{-\pi}^{\pi} \frac{v^2(t)dt}{m\varepsilon + u(t) - m\varepsilon - u_{min}} =$$

$$= -u_{min} - \alpha^2 \int_{-\pi}^{\pi} \frac{v^2(t)dt}{u(t) - u_{min}}$$

munosabatlarga ega bo‘lamiz.

Endi $A_\alpha(m)$ operatorning $m\varepsilon + u_{min}$ dan chapda yotuvchi xos qiymatlar mavjudligi masalasini o‘rganamiz.

1-hol. Agar $u_{min} \geq 0$ bo‘lsa, u holda

$$\int_{-\pi}^{\pi} \frac{v^2(t)dt}{u(t) - u_{min}}$$

Integralning qiymati chekli yoki cheksiz bo‘lishidan qat’iy nazar ixtiyoriy $\alpha > 0$ soni uchun $\Delta_\alpha(m; m\varepsilon + u) < 0$ bo‘ladi.3-xossaga ko‘ra $A_\alpha(m)$ operator ixtiyoriy $\alpha > 0$ son uchun $(-\infty\,; m\varepsilon + u_{min})$ oraliqda yotuvchi yagona xos qiymatga ega.

Faraz qilaylik $u_{min} < 0$ bo‘lsin. Bunda

$$\int_{-\pi}^{\pi} \frac{v^2(t)dt}{u(t) - u_{min}}$$

integral chekli yoki cheksiz bo‘ladigan hollar farqlanadi.

Agar

$$\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u}$$

integral cheksiz (ya'ni uzoqlashuvchi) bo'lsa, u holda

$$\Delta_\alpha(m; m\varepsilon+u_{min}) = -\infty$$

bo'ladi. $\lim_{z\to-\infty}\Delta_\alpha(m;\ z) = +\infty$ ekanligini hamda $\Delta_\alpha(m;\cdot)$ funksiyaning monoton kamayuvchi ekanligini inobatga olsak, yagona $z_0\in(-\infty; m\varepsilon+u_{min})$ soni topilib , $\Delta_\alpha(m;z_0)=0$ tenglik bajariladi. 1-xossaga ko'ra ixtiyoriy $\alpha>0$ soni uchun $A_\alpha(m)$ operator $(-\infty; m\varepsilon+u_{min})$ oraliqda yotuvchi yagona xos qiymatga ega.

Faraz qilaylik,

$$\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}}$$

integral chekli bo'lsin. U holda

$$\Delta_\alpha(m; m\varepsilon+u) \geq 0 ,$$

ya'ni

$$\Delta_\alpha(m; m\varepsilon+u) = -u-\alpha^2\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}} \geq 0$$

bo'lsin.

$$\alpha^2\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u} \leq -u_{min};$$

$$\alpha^2 \leq -u_{min}[\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}}]^{-1};$$

$$\alpha \leq \sqrt{-u_{min}}[\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}}]^{-\frac{1}{2}}$$

munosabatlar o'rinli bo'ladi. α parametrning quyidagi kritik qiymatini kiritamiz:

$$\alpha_0 := \sqrt{-u_{min}}[\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}}]^{-\frac{1}{2}}$$

Shunday qilib quyidagi xossa o'rinli ekan.

5-xossa. a) *Faraz qilaylik* $u_{min}\geq 0$ *bo'lsin , u holda ixtiyoriy* $\alpha>0$ *son uchun* $A_\alpha(m)$ *operator* $(-\infty; m\varepsilon+u_{min})$ *oraliqda yotuvchi yagona xos qiymatga ega.*

b) *Faraz qilaylik* $u_{min}<0$ *bo'lsin,*

b.1) *Agar*

$$\int_{-\pi}^{\pi}\frac{v^2(t)dt}{u(t)-u_{min}} \quad (1)$$

integral uzoqlashuvchi bo'lsa, u holda u holda ixtiyoriy $\alpha>0$ *son uchun* $A_\alpha(m)$ *operator* $(-\infty; m\varepsilon+u_{min})$ *oraliqda yotuvchi yagona xos qiymatga ega.*

b.2) *Faraz qilaylik* (1)- *integral yaqinlashuvchi bo'lsin, u holda ixtiyoriy* $\alpha\in(0;\alpha_0]$ *soni uchun* $A_\alpha(m)$ *operator* $(-\infty; m\varepsilon+u_{min})$ *oraliqda yotuvchi xos* *qiymatlarga ega emas. Barcha $\alpha > \alpha_0$ son uchun $A_\alpha(m)$ operator (-∞ ; $m\varepsilon + u_{min}$) oraliqda yotuvchi xos qiymatga ega.*

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