Calculation of some improper integrals by Euler integrals

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Abstract: To resolve several challenging applications in many scientific domains, general formulas of improper integrals are provided and established for use in this article. The suggested functions can be considered generators for new improper integrals with precise solutions, without requiring complex computations. New criteria for handling improper integrals are using Gamma and Betta function properties.

Keywords: improper integrals, power series, analytic function, Euler integral, Gamma function, Betta function

This

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad (a>0, b>0) \quad (1)$$

improper integral is called the Betta function or Euler integral of type 1 and is denoted as $B(a,b)$, namely

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx . \quad (2)$$

For a function under the integral:

1) when $a<0, b\geq 0$, point $x=0$;

2) when $a\geq 1, b<1$, point $x=1$;

3) when $a<1, b<1$, point $x=0$ and $x=1$ are special points.

Hence, the integral (1) is a improper integral dependent on the parameter.

The Betta function has the following properties.

Property 1. Integral (1) - convergent in the set.

$$M=\{(a;b)\in R^2 : a\in(0;+\infty), b\in(0;+\infty)\}$$

Property 2. Integral (1) on $M_0=\{(a;b)\in R^2 : a\in[a_0;+\infty), b\in[b_0;+\infty)\}, a_0>0, b_0>0$ uniformly converging on the set, but unevenly converging on the set M.

Property 3. The function $B(a,b)$ is a continuous function on the set M.

Property 4. $\forall(a,b)\in M$ for $B(a,b)=B(b,a)$ (symmetry) be.

Property 5. $B(a,b)$ the function is also expressed as:

$$B(a,b)=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt$$

Prove:

$$B(a;b)=\int_0^1 x^{a-1}\cdot(1-x)^{b-1}dx=\left[\begin{array}{l}\frac{x}{1-x}=t\\ x=t-tx\\ x+tx=t\\ x\cdot(1+t)=t\\ x=\frac{t}{1+t}\\ dx=\frac{t+1-t}{(1+t)^2}dt=\frac{dt}{(1+t)^2}\\ 1-x=1-\frac{t}{1+t}=\frac{1}{1+t}\end{array}\right]$$

$$\int_0^\infty\left(\frac{t}{1+t}\right)^{a-1}\cdot\left(\frac{1}{1+t}\right)^{b-1}\cdot\frac{dt}{(1+t)^2}=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a-1}}\cdot\frac{1}{(t+t)^{b-1}}\cdot\frac{dt}{(1+t)^2}=$$

$$=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b-2+2}}dt=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

proved.

Property 6. $\forall(a,b)\in M_1=\{(a,b)\in R^2 : a\in(0;+\infty), b\in(1;+\infty)$ for

$$B(a,b)=\frac{b-1}{a+b-1}\cdot B(a,b-1)$$

Property 7. $B(a,1-a)=\frac{\pi}{\sin a\pi}(0<a<1)$, in the case $a=\frac{1}{2}$, $B=\left(\frac{1}{2},\frac{1}{2}\right)=\pi$

Gamma function and its properties.

This

$$\int_0^{+\infty}x^{a-1}e^{-x}dx \qquad (a>0) \tag{3}$$

improper integral is called the Gamma function or Euler integral of type 2 and is denoted as

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx \qquad (a>0) \tag{4}$$

For a function under the integral:

1) When $a<1$, $x=0$ point - a special point;

2) When $a>1$, (3) integral is convergent;

3) When $a\le 0$, (3) integral is divergent;

The Gamma function has the properties below.

Property 1.

$$\Gamma(a)=\lim_{n\to\infty} n^a \frac{1\cdot 2\cdot 3\cdots(n-1)}{a(a=1)\cdots(a+n+1)}. \quad (5)$$

the formula (5) is called the Euler-Gauss formula.

Property 2. (3) integral in the interval $\forall a\in[a_0,b_0]\,(0<a_0<b_0<+\infty)$ uniformly convergent, but unevenly converging on $a\in(0,+\infty)$.

Property 3. Function $\Gamma(a)$ in the interval $(0;+\infty)$ is continuous and has continuous derivatives of all orders, namely

$$\Gamma^{(n)}(a)=\int_0^{+\infty} x^{a-1}e^{x}(\ln x)^n dx \quad (n\in N).$$

Property 4. $\Gamma(a+1)=a\Gamma(a) \quad (a>0).$

Property 5. $\Gamma(n+1)=n!$

Property 6. $B(a,b)\equiv\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$

Property 7. $\Gamma(a)\Gamma(1-a)=B(a,1-a)=\dfrac{\pi}{\sin a\pi}, \quad 0<a<1.$

Property 8. $\Gamma(n+\dfrac{1}{2})=\dfrac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad n\in N.$

Property 9. The Lejandr formula: $\Gamma(a)\Gamma(a+\dfrac{1}{2})=\dfrac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a).$

Now let's look at several examples of calculating some improper integrals using the Betta and Gamma functions.

Problem 1. Calculate the improper integral below using the Euler integral.

$$\int_0^2 \frac{dx}{\sqrt[3]{x^2(2-x)}}$$

Solution. By replacing $\dfrac{x}{2-x}=t$ in this integral, we obtain the following integral and calculate the value.

$$\left[\begin{array}{l} \dfrac{x}{2-x}=t, \\ x=2t-tx, \\ x\cdot(1+t)=2t \\ x=\dfrac{2t}{1+t} \\ dx=\dfrac{2\cdot(1+t)}{(1+t)^2}dt=\dfrac{2}{(1+t)^2}dt \end{array}\right] \Rightarrow \int_0^{\infty}\frac{2dt}{(1+t)^2\cdot\sqrt[3]{(\frac{2t}{1+t})^2\cdot(2-\frac{2t}{1+t})}}=\int_0^{\infty}\frac{2dt}{(1+t)^2\cdot\sqrt[3]{\cdot\frac{4t^2}{(1+t)^2}\cdot\frac{2}{1+t}}}=$$

$$=\int_0^{\infty}\frac{2dt}{(1+t)^2\cdot\frac{2}{t+1}\cdot t^{\frac{2}{3}}}=[by\ substuiting]=\int_0^{\infty}\frac{t^{\frac{-2}{3}}}{1+t}dt=\int_0^{\infty}\frac{t^{1-\frac{2}{3}-1}}{1+t}dt=$$

$$=\int_0^{\infty}\frac{t^{\frac{1}{3}-1}}{(1+t)^{\frac{1}{3}+\frac{2}{3}}}dt=B(\frac{1}{3};\frac{2}{3})=[according\ to\ 7th\ property\ of\ Betta\ function]=\frac{\pi}{\sin\frac{\pi}{3}}=\frac{\pi}{\frac{\sqrt{3}}{2}}=\frac{2\pi}{\sqrt{3}}$$

Problem 2. Using the Euler integral, calculate $\int_0^{\infty}x^4\cdot e^{-x^2}dx$

Solution. By replacing $x^2=t$ in this integral, we obtain the following integral and calculate the value

$$\begin{bmatrix} x^2=t, \\ x\to 0\ \ da\ \ t\to 0,\ \ x\to\infty\ \ da\ \ t\to\infty \\ x=\sqrt{t} \\ dx=\frac{dt}{2\sqrt{t}} \\ x^4=t^2 \end{bmatrix}\Rightarrow$$

$$\Rightarrow\int_0^{\infty}t^2\cdot e^{-t}\cdot\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^{\infty}t^{\frac{3}{2}}\cdot e^{-t}dt=\frac{1}{2}\int_0^{\infty}t^{\frac{5}{2}-1}\cdot e^{-t}dt=\frac{1}{2}\Gamma\left(\frac{5}{2}\right)=$$

$$=[according\ to\ 8th\ property\ of\ Gamma\ function]=\frac{1}{2}\Gamma\left(2+\frac{1}{2}\right)=$$

$$=\frac{1}{2}\cdot\frac{(2\cdot 2-1)!!}{2^2}\cdot\sqrt{\pi}=\frac{1}{2}\cdot\frac{3!!}{4}\cdot\sqrt{\pi}=\frac{3\sqrt{\pi}}{8}$$

Problem 3. Using the Euler integral, calculate it:

$$\int_0^{\frac{\pi}{2}}\sin^4 x\cos^6 xdx=\begin{bmatrix} \sin x=\sqrt{x}, & x\to 0\ \ da\ \ t\to 0 \\ x=\arcsin\sqrt{t}, & x\to\frac{\pi}{2}\ \ da\ \ t\to 1 \\ dx=\frac{dt}{\sqrt{1-t}\cdot 2\sqrt{t}} & \end{bmatrix}=\int_0^1 t^2(1-t)^3\frac{dt}{\sqrt{1-t}\cdot 2\sqrt{t}}=$$

$$=[simplifying]=\frac{1}{2}\int_0^1 t^{\frac{3}{2}}\cdot(1-t)^{\frac{5}{2}}dt=\frac{1}{2}\int_0^1 t^{1+\frac{3}{2}-1}\cdot(1-t)^{1+\frac{5}{2}-1}dt=\frac{1}{2}\int_0^1 t^{\frac{5}{2}-1}\cdot(1-t)^{\frac{7}{2}-1}dt=$$

$$=\frac{1}{2}B(\frac{5}{2};\frac{7}{2})=\frac{1}{2}\cdot\frac{\Gamma(\frac{5}{2})\cdot\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2}+\frac{7}{2})}=\frac{1}{2}\cdot\frac{\Gamma(\frac{5}{2})\cdot\Gamma(\frac{7}{2})}{\Gamma(6)}=\frac{1}{2}\cdot\frac{\Gamma(2+\frac{1}{2})\cdot\Gamma(3+\frac{1}{2})}{5!}=$$

$$=\frac{1}{2}\cdot\frac{\frac{(2\cdot 2-1)!!}{2^2}\cdot\sqrt{\pi}\cdot\frac{(2\cdot 3-1)!!}{2^3}\cdot\sqrt{\pi}}{5!}=\frac{1}{2}\cdot\frac{\frac{3!!}{4}\cdot\sqrt{\pi}\cdot\frac{5!!}{8}\cdot\sqrt{\pi}}{120}=\frac{1\cdot 3\cdot 1\cdot 3\cdot 5\cdot\pi}{64\cdot 120}=\frac{3\pi}{64\cdot 8}=\frac{3\pi}{512}$$

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