

Methods of Solving Differential Equations

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Abstract: A variety of physical processes, including heat conduction, wave propagation, fluid movement, and quantum physics, are fundamentally described by partial differential equations, or PDEs. Multiple variable functions and their partial derivatives are involved in these equations. Based on the number of independent variables, the type of boundary and initial conditions, and the nature of the equations (linear or nonlinear), PDEs can be categorized.

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When solving PDEs, it is common to need to identify a function that satisfies both the differential equation and the accompanying initial or boundary conditions. This can be difficult due to the complexities of these equations and the wide range of conceivable situations. There are two types of approaches for solving PDEs: analytical and numerical procedures. Although analytical techniques yield precise answers, they are frequently restricted to issues that are simpler and more idealized. Numerical methods, on the other hand, offer approximate solutions and are more flexible, making them suitable for complex, real-world problems.

PDEs are crucial in modeling systems across different scientific and engineering disciplines:

- **Physics:** Describing electromagnetic fields, quantum mechanics, and general relativity.
- **Engineering:** Modeling stress and strain in materials, fluid dynamics, and heat transfer.
- **Biology:** Representing processes like diffusion, population dynamics, and neural activity.
- **Finance:** Modeling options pricing and risk management in financial markets.

Given their importance, understanding and solving PDEs is a key skill in these fields. The methods for solving PDEs can be categorized into two main types: analytics and numerical methods.

Analytical Methods

Analytical methods aim to find exact solutions to PDEs. These methods leverage mathematical techniques to derive a closed-form solution that satisfies the PDE and the

initial or boundary conditions. Some commonly used analytical methods are Separation of Variables, Method of Characteristics, Transform Methods.

I. Separation of Variables

Separation of Variables involves assuming that the solution can be written as a product of functions, each depending on a single variable. This method is particularly effective for linear PDEs with homogeneous boundary conditions. The primary advantage of this method is its simplicity and direct approach, making it useful for problems with well-defined boundary conditions. For instance, we consider this mixed problem of one-dimensional heat equation:

$$\begin{cases} u_t = \alpha u_{xx} \\ u(x, 0) = f(x) \\ u(0, t) = u(L, t) = 0 \end{cases}$$

Used steps to find solution of above problem ([1]-[3]) include:

1. Assume a solution of the form:

$$u(x, t) = X(x)T(t).$$

2. Substitute into the heat equation:

$$X(x)T'(t) = \alpha X''(x)T(t).$$

3. Separate variables:

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

4. Solve the spatial part:

$$X''(x) + \lambda X(x) = 0$$

with boundary conditions $X(0) = X(L) = 0$. The solutions are:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

5. Solve the temporal part:

$$T'(t) + \alpha\lambda_n T(t) = 0.$$

The solutions are:

$$T_n(t) = e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

6. Combine the solutions:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}.$$

7. Determine the coefficients B_n using the initial condition:

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

II. Method of Characteristics.

The Method of Characteristics transforms a PDE into a set of ordinary differential equations (ODEs) along characteristic curves. This method is particularly useful for first-order PDEs, allowing us to reduce the problem to solving simpler ODEs. The primary advantage is its applicability to non-linear PDEs and its ability to provide insights into the behavior of solutions along specific paths or curves in the domain. For example, we consider this transport equation with initial condition:

$$u_t + cu_x = 0$$

$$u(x, 0) = g(x)$$

Steps applied to solve the problem described above include:

1. Identify the characteristic equations:

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0.$$

2. Solve the characteristic equations:

$$x = ct + x_0.$$

Since u is constant along characteristics:

$$u(x, t) = g(x_0).$$

3. Express x_0 in terms of x and t : $x_0 = x - ct$. Thus, the solution is:

$$u(x, t) = g(x - ct).$$

III. Transform Methods

Transform methods convert PDEs into simpler forms by transforming the variables. The Fourier Transform and Laplace Transform are commonly used to convert PDEs into algebraic equations or ordinary differential equations in the transform domain. These methods are particularly useful for linear PDEs and problems with infinite or semi-infinite domains. The primary advantage is their ability to handle complex boundary conditions and transform convoluted differential operators into simpler multiplication operators. For example, we consider the following heat equation in an infinite domain:

$$u_t = \alpha u_{xx}$$

$$u(x, 0) = f(x)$$

Steps used to solve this problem are:

1. Apply the Fourier Transform ([1]-[3]) to both sides:

$$\hat{u}_t = -\alpha \omega^2 \hat{u}.$$

2. Solve the resulting ODE:

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-\alpha\omega^2 t}.$$

3. Apply the Inverse Fourier Transform ([1]):

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-\alpha\omega^2 t} e^{i\omega x} d\omega.$$

Numerical Methods

Numerical methods are essential for solving PDEs when analytical solutions are difficult or impossible to obtain. These methods approximate the solution by discretizing the domain and solving the resulting system of equations. Common numerical methods include the Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM).

I. Finite Difference Method (FDM)

The Finite Difference Method approximates derivatives by finite differences and solves the resulting algebraic equations. It involves discretizing the continuous domain into a grid and replacing the continuous derivatives with discrete approximations. This method is straightforward and easy to implement, especially for simple geometries and uniform grids. The primary advantage is its simplicity and ease of implementation for time-dependent problems. For example, heat equation using explicit scheme.

Let's consider the heat equation $u_t = \alpha u_{xx}$ with boundary condition $u(0, t) = 0$, $u(L, t) = 0$, and initial condition $u(x, 0) = f(x)$.

Steps used in FDM ([4]) include:

1. Discretize the domain:

$$x_i = i\Delta x \text{ for } i = 0, 1, 2, \dots, N$$

$$t^n = n\Delta t$$

2. Approximate derivatives using finite differences:

$$u_t \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}, \quad u_{xx} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

3. Substitute into the heat equation:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}.$$

4. Solve for u_i^{n+1} :

$$u_i^{n+1} = u_i^n + \frac{\alpha\Delta t}{(\Delta x)^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Let $r = \frac{\alpha\Delta t}{(\Delta x)^2}$:

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

5. Apply initial and boundary conditions:

$$u_i^0 = f(x_i), \quad u_0^n = 0, \quad u_N^n = 0.$$

6. Iterate over time steps. Update u_i^{n+1} for each i and n using the scheme.

Suppose $\alpha = 1, L = 1$, and we discretize with $\Delta x = 0.1$ va $\Delta t = 0.005$. The initial condition is $u(x, 0) = \sin(\pi x)$. Then initialization is

$$u_i^0 = \sin(\pi x_i), \quad r = \frac{1 \times 0.005}{(0.1)^2} = 0.5.$$

We iterate over time steps:

$$u_i^{n+1} = u_i^n + 0.5(u_{i+1}^n - 2u_i^n + u_{i-1}^n).$$

By iterating this process, we numerically solve the heat equation.

II. Finite Element Method (FEM)

The Finite Element Method divides the domain into smaller subdomains (elements) and uses piecewise polynomial functions to approximate the solution. It involves formulating the PDE as a variational problem and approximating the solution using basis functions defined over the elements. The primary advantage of FEM is its flexibility in handling complex geometries and boundary conditions. It is widely used in engineering applications for its robustness and accuracy. For instance, we consider Poisson's equation

$$-\Delta u = f$$

in a domain Ω with boundary condition $u = 0$ on $\partial\Omega$.

Steps to find solution with FEM ([5]) are:

1. Formulate the weak form. Multiply by a test function v and integrate:

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f v \, d\Omega$$

2. Discretize the domain. Divide Ω into finite elements and approximate u and v by piecewise linear functions.

3. Assemble the system. Form the stiffness matrix K and load vector F from the integrals.

4. Solve the linear system:

$$KU = F, \text{ where } U \text{ is the vector of nodal values of } u.$$

For a simple 1D example on $[0, 1]$, divide the interval into N elements with nodes $x_i = i\Delta x$. The weak form will be

$$\int_0^1 u'v' \, dx = \int_0^1 f v \, dx$$

Then we use linear basis functions ϕ_i such that $\phi_i(x_j) = \delta_{ij}$. After we find stiffness matrix and load vector

$$K_{ij} = \int_0^1 \phi_i' \phi_j' dx, \quad F_i = \int_0^1 f \phi_i dx$$

Then we solve the system $KU = F$. By solving this system, we obtain the approximate solution $u(x)$.

Conclusion

Solving IVPs and BVPs for PDEs involves a variety of methods, each suitable for different types of problems. Analytical methods like Separation of Variables and the Method of Characteristics are powerful for certain classes of PDEs, while numerical methods like the Finite Difference Method and Finite Element Method provide flexibility for more complex or irregular domains. Understanding these methods and their applications is crucial for effectively tackling PDEs in various scientific and engineering fields.

References

1. Walter A. Strauss, Partial Differential Equations: An Introduction(2nd ed.), Wiley, 2007.
2. Lawrence C. Evans, Partial Differential Equations, American Mathematical Society, 2010.
3. Claes Johnson, Numerical Solution of Partial Differential Equations by the Finite Element Method, Dover Publications, 1987.
4. Randall J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations, SIAM, 2007.
5. Steven C. Chapra and Raymond P. Canale, Numerical Methods for Engineers, McGraw-Hill Education, 2015.
6. Daryl L. Logan, A First Course in the Finite Element Method, Cengage Learning, 2016.
7. Kuchkorov E.I, Jumaeva S.F, A one-dimensional fractional diffusion equation with discontinuous diffusion coefficient, Mathematics, Mechanics and Intellectual Technologies Tashkent, 2023.
8. Kuchkorov E., Jumaeva Sh. Mixed-type problem for the time-fractional diffusion equation with a discontinuous coefficient, Differensial tenglamalarning zamonaviy muammolari va ularning tatbiqlari