Application of the Drazin Inverse on the Solution of the first-order linear singular differential equations

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Abstract: The Drazin Inverse is a generalized form of inverse that shares similar properties with a square matrix; hence, the Drazin inverse is only defined for a square matrix. The Drazin inverse has numerous applications in solving singular differential equations, Markov chains, and iterative methods in numerical analysis. In the case where the system $Ax = b$ involves a matrix A that is invertible, if matrix A is singular or non-invertible "Certainly! The corrected text is: "inverse, denote"e, the Drazin inverse is utilized to solve the system above. The research aim is to utilize the Drazin inverse in solving first-order singular linear differential equations. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathbf{n} \times \mathbf{n}$ matrices, $\boldsymbol{f}$ a vector-valued function. $\boldsymbol{A}$ and $\boldsymbol{B}$ may both be singular. The differential equation $\boldsymbol{Ax}' + \boldsymbol{Bx} = \boldsymbol{f}$ is examined using the theory of the Drazin inverse. $\boldsymbol{A}$ closed form expression for all solutions of the differential equation is provided when the equation has unique solutions for consistent initial conditions. This is a review article and the results show that to solve single linear differential equations, the inverse of Drazin is the best possible way to solve this type of differential equations.

Keywords: Drazine inverse, exponential matrix, generalized inverse, Linear equations

INTRODUCTION

The Drazin inverse was named by Michael P. Drazin. It is a generalized inverse that has the same properties as a matrix's normal inverse and is therefore defined only for a square matrix.

The Durbin inverse has various applications in solving singular differential equations, Markov chains, and iterative methods in numerical analysis.

Suppose we have the differential equation $x'(t) + Ax(t) = f$, where f is a constant vector. If matrix A is not invertible, solving this system will encounter difficulties. Therefore, for solving this system, the Drazin inverse matrix of matrix A is utilized. In such cases, the significance of utilizing the Drazin inverse matrix is remarkably high.

In 1983, Cumblet, based on research conducted on singular differential equations, sought to find an explicit representation of the Durbin inverse for block matrices $\begin{bmatrix} A & B \\ -I & 0 \end{bmatrix}$, where matrices E and F are square, and if $AB = BA$ is satisfied. The solution to the singular differential equation $Ax't$+Bxt=0 can be obtained using this.

Methods and Study Site: This article is a review study that was done by searching databases such as Google Scholar, web of Since, scopus and using the keywords Linear equations, generalized inverse, Drazine inverse and exponential matrix, the abstracts of 57 articles were studied, and only 11 The article related to the topic was selected and reviewed.

Drazin reverse: Definition 1: Let $\mathrm{A} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ such that $\mathrm{And(A)} = 1$. A matrix $\mathrm{X} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ is said to be related to A if it satisfies the following conditions:

$$\mathrm{XAX} = \mathrm{X}, \mathrm{AX} = \mathrm{XA}, \mathrm{A^kXA} = \mathrm{A^k}, \mathrm{k} \geq 0 \ (1)$$

If these conditions hold, X is called the Drazin inverse of A and is denoted by $\mathrm{A^D}$. When $And(A) = 1$, the Drazin inverse $\mathrm{A^D}$ has an inverse concerning its length, known as the group inverse, denoted by $\mathrm{A^\#}$. If $And(A) = 0$, then the matrix A is invertible, and its Drazin inverse is simply $\mathrm{A^{-1}}$. When $\mathrm{A^\#}$exists, the following equations are satisfied [1]:

$$\mathrm{AA^\#A} = \mathrm{A}, \mathrm{A^\#AA^\#} = \mathrm{A^\#} \ and \ \mathrm{AA^\#} = \mathrm{A^\#A}.$$

Some basic properties of Drazin inverses

Suppose $\mathrm{A} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ and $\mathrm{And(A)} = \mathrm{k} > 0$. then[2]:

1. For each real and positive number $A^{p+1}A^D = A^D$
2. $(A^T)^D = (A^D)^T$.
3. $(A^D)^D = A \Leftrightarrow Ind(A) \leq 1$

Theorem 1: Let $\mathrm{A} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$and $\mathrm{And\ (A)} = \mathrm{k} > 0$. then, there exists an invertible matrix P such that

$$\mathrm{A} = \mathrm{P}\begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{N} \end{bmatrix}\mathrm{P^{-1}}.$$

where C is an invertible matrix with $\mathrm{rank(C)} = \mathrm{rank}(\mathrm{A^k})$ and N is a nilpotent matrix of order k [3]

Theorem 2 Suppose C, P and N are matrices that apply to the conditions of Theorem1,[3]. then

$$\mathrm{A^D} = \mathrm{P}\begin{bmatrix} \mathrm{C^{-1}} & 0 \\ 0 & 0 \end{bmatrix}\mathrm{P^{-1}} \ (2)$$

the exponential matrix in solving linear differential equations: The exponential matrix method is a useful technique for solving systems of linear differential equations. To apply this method, we start with a system of differential equations which can be written in matrix form as[4]:

$$\mathrm{x'(t)} = \mathrm{Ax(t)}$$

The general solution to this system is given by:

$$x(t) = e^{tA}C.$$

In this expression the above relation, $C = (c_1, c_2, c_3, \ldots, c_n)^T$ is a vector containing n components. For an initial value problem, the components of C are determined by the initial conditions. Thus, the solution can be written as:

$$x(t) = e^{tA}x_0 \; x_0 = X(t = t_0).$$

Example 1: Using the exponential matrix method, we want to solve the following system of equations[4].

$$\begin{cases} \dfrac{dx}{dt} = 4x \\ \dfrac{dy}{dt} = x + 4y \end{cases}$$

The characteristic equation of the above system and its eigenvalues are equal to:

$$\det(A - \lambda_I) = \begin{vmatrix} 4-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

Therefore, only one repeated eigenvalue was found for this system$(4\lambda, 4)$, the eigenvector $V_1 = (v_{11}, v_{21})^T$is equal to

$$\begin{vmatrix} 4-4 & 0 \\ 1 & 4-4 \end{vmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0, \Rightarrow 1 \cdot v_{11} + 0 \cdot v_{21} = 0$$

The above equation tells us that v_{21} is equal to zero, on the other hand, the value of v_{21} can be any arbitrary number and finally the special vector v_1 is equal to:

$$v_1 = (0,1)^T$$

The second independent vector which is considered as $V_2 = (v_{12}, v_{22})^T$ is also obtained by using the equations mentioned below.

$$(A - \lambda I)v_2 = v_1, \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \Rightarrow \begin{cases} 0 \cdot v_{12} + 0 \cdot v_{22} = 0 \\ 1 \cdot v_{12} + 0 \cdot v_{22} = 1' \end{cases}$$

In the above equation, the value of v_{22} can be any number. Finally, for the sake of simplicity, the following values are considered:

$$v_{22} = 0, v_{11} = 0$$

Therefore, finally, the vector v_2 is equal to:

$$v_2 = (1,0)^T$$

Now using the base vector, the matrix H will be equal to:

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Also, the H^{-1} inverse vector is equal to:

$$\Delta(H) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, H^{-1} = \frac{1}{\Delta(H)} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^T$$

In this matrix, the H_{ij} values are the cofactors of the H matrix. After calculating the H matrix, it is equal to:

$$H^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{T} = (-1)\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In some interesting occasions, two functions H and H^{-1} are equal to each other, this happens when the second power of a matrix is equal to:

$$H^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Jordan's form or J corresponding to matrix A is also equal to:

$$J = H^{-1}AH = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

By obtaining the Jordan form, the transformation matrix will also be equal to:

$$e^{tj} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix} = e^{4t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

As a result, the exponential matrix is equal to:

$$e^{tA} = He^{tj}H^{-1} = e^{4t}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e^{4t}\begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e^{4t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

Therefore, finally, the general answer of the system is equal to[5]:

$$X(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{tA}C = e^{4t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

First-order linear singular differential equations

1. We consider the first-order linear differential equation as follows[3]

$$x'(t) + Ax(t) = f \; (3)$$

With f is a constant vector. The general solution of equation (3) is:

$$x(t) = \left[e^{-At}\int_a^t e^{At}\,dt\right] f \; (4)$$

With $a \in \mathbb{R}$ let A is non-singular then

$$\int e^{At}\,dt = A^{-1}e^{At} + G, G \in \mathbb{R}^{n\times n}$$

If A singular then the problem becomes somewhat complex, it can use the following theorem with the help of the Drazin inverse to solve the problem.

Theorem 3: if $A \in \mathbb{R}^{n\times n}, Ind(A) = k$ then[6]:

$$\int e^{At}\,dt = A^{D}e^{At} + (I - AA^{D})t\left[I + \frac{At}{2!} + \frac{A^2t^2}{3!} + \cdots + \frac{A^{k-1}t^{k-1}}{k!}\right] + G, G \in \mathbb{R}^{n\times n} \; (5)$$

Example 2: let $x'(t) + Ax(t) = f$ with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A^{D} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering that $Ind(A) = 1$ the solution of the equation is as follows:

$$x(t) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{t}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{t}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

2. Study of the Differential Equation $Ax' + Bx = f$ When $AB = BA$

This section begins the examination of the differential equation

$$Ax' + Bxf \ (6)$$

when A and B commute($B = BA$). If A is nonsingular, equation (6) can be expressed in a form similar to (3). We consider cases where both A and B may be singular[6]. While methods for solving (6) are well-established, such as those described by Gantmacher[6], our approach aims to provide closed-form solutions without relying on elementary divisors or the canonical form for a pencil. Additionally, we avoid introducing solutions for auxiliary equations that do not satisfy the original equation, a common issue when using inverses other than the Drazin inverse.

The associated homogeneous equation is:

$$Ax' + Bx = 0 \ (7)$$

We assume that A and B commute in this section. It will be shown that if consistent initial conditions uniquely determine solutions, equation (6) can be reduced to a case where A and B commute.

Let $x_1 = A^D Ax$ and and $x_2 = (I - A^D A)x$. Then equation (6) transforms into:

$$(C + N)(x'_1 + x'_2) + B(x_1 + x_1) = f.$$

By Multiplying first by $C^D C$ and then by$(I - C^D C)$, we get that (7) is equivalent to

$$Cx'_1 + Bx'_1 = f_1 \ (8)$$

and

$$Nx'_2 + Bx_2 = f_2 \ (9)$$

Where $f_1 = C^D Cf \ and \ f_1 = (I - C^D C)f$. The Equation $Cx'_1 + Bx'_1 = f_1$ can be rewritten as

$$x'_1 + C^D Bx_1 = C^D f, (10)$$

which matches the form of (3) Hence, this equation has a unique solution for all initial conditions in $\mathbb{R}(A^D A)$. However, equation (9) may or may not have nontrivial solutions, and these solutions, if they exist, are not necessarily uniquely determined by initial conditions. Before providing an example, we will draw some conclusions from the equation (10).

Theorem 4. Suppose that A and B commute. Then $y = e^{-A^D Bt} AA^D q$ is a solution to the differential equation $Ax' + Bx = 0$ for every column vector q.

Corollary1. If A and B commute and $A^D Af = f$, then $y = e^{-A^D Bt} \int e^{A^D Bt} f(t)dt$ is a particular solution of $Ax' + Bx = f$.

It is worth noting that in Theorem 4 and Corollary 1, the assumption that A and B commute can be replaced by the weaker assumption that A^DA and B commute. Details of this can be found in the literature for those interested.

Now let us consider a special case of (7). Since it is typically the nilpotent parts that cause difficulties, we will take A and B to be nilpotent[3].

Example 3. Consider the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, the equation (7) can be written as:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which simplifies to $x'_2 + x_2 + x_3 = 0$. In this case, x_1 and x_2 can be arbitrary, even if initial conditions are imposed. Note that $AB = BA$. The following lemma will be fundamental for the subsequent discussions.

Lemma1. Suppose that matrices A and B commute ($AB = BA$) and the intersection of the null spaces of A and B contains only the zero vector. Then the following holds:

The expression$(I - A^DA)BB^D$ is equal to $(I - A^DA)$

Theorem 5**:** Assume that matrices A and B commute with each other, and the intersection of their null spaces contains only the zero vector. Then, the general solution to the differential equation

$Ax' + Bx = 0$ is given by:

$$x = e^{-A^DBt}AA^Dq$$

where q is an arbitrary vector in $\mathbb{C}^n$.

Example 4. Consider the system $Ax' + Bx = 0$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; and \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $A^D = 0$ and the intersection of the null spaces of A and B is only the zero vector $N(A) \cap N(B) = 0$, Theorem 5 tells us that the system has only the trivial solution[3]. Let

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Then E is a (1,2)-inverse of A. the expression$e^{-EBt}EAq = e^{-Et}EAq$ is not identically zero for all q, so it does not provide a solution to $Ax' + Bx = 0$.

We now provide a particular solution to the equation $Ax' + Bx = 0$when A and B commute and the intersection of their null spaces is the zero vector.

As usual, $f^{(n)} = \frac{d^nf}{dt^n}$ denotes the n-th derivative of f with respect to t.

Theorem 6. Suppose that matrices A and B commute$(\mathrm{AB} = \mathrm{BA})$ and the intersection of their null spaces is the zero vector $N(A) \cap N(B) = \{0\}$. Let $\mathrm{Ind(A)} = \mathrm{k}$ If f is a vector-valued function that is continuously differentiable k times, then the non-homogeneous linear differential equation:

$$Ax' + Bx = f \ (6)$$

is consistent. A particular solution to this equation is given by:

$$x = e^{-A^D Bt} \int_a^t e^{A^D Bs} f(s) ds + (I - AA^D) \sum_{n=0}^{k-1} (-1)^n (AB^D)^n B^D f^{(n)} \ (11)$$

Where a is an arbitrary constant.

Theorem 7. Suppose that matrices A and B commute$(\mathrm{AB} = \mathrm{BA})$ and the intersection of their null spaces is the zero vector $(\mathrm{N(A)} \cap \mathrm{N(B)} = \{0\})$. Then the general solution to the non-homogeneous differential equation:

$$Ax' + Bx = f \ (6)$$

is given by:

$$x = e^{-A^D Bt} A^D q + A^D e^{-A^D Bt} \int_a^t e^{A^D Bs} f(s) ds + (I - AA^D) \sum_{n=0}^{k-1} (-1)^n (AB^D)^n B^D f^{(n)} \ (12)$$

where q is an arbitrary constant vector, $Ind\ (A) = k$ and a is an arbitrary constant.

As an immediate corollary of Theorem 7, we obtain a characterization of consistent initial conditions when $AB = BA$ and $N(A) \cap N(B) = \{0\}$.

Corollary 2. Suppose that matrices A and B commute $(\mathrm{AB} = \mathrm{BA})$ and the intersection of their null spaces is only the zero vector $(\mathrm{N(A)} \cap \mathrm{N(B)} = \{0\})$. Then there exists a solution to the differential equation $to\ Ax' + Bx = f$ with the initial condition $x(0) = x_0$ if and only if $x(0) = x_0$ is of the form:

$$x_0 = A^D q + (I - AA^D) \sum_{n=0}^{k-1} (-1)^n (AB^D)^n B^D f^{(n)}(0)$$

for some vector q. Furthermore, the solution is unique. Specifically, if$f$ is identically zero, then the condition $A^D A x_0 = x_0$ characterizes consistent initial conditions. Corollary 2 can be used for initial conditions at nonzero values by performing a change of variables. Note that if B is invertible in the given equation, then Theorem 7 can be applied to the transformed equation $B^{-1}Ax' + x = B^{-1}f$, and the techniques from the next section are not necessary.

4. The equation $Ax' + Bx = f$. In this section, we will determine the necessary and sufficient conditions for ensuring the uniqueness of solutions to the equation $Ax' +$

$Bx = f$ We will then use Theorem 7 to find the general solution in cases where a unique solution exists. The forthcoming lemma will be crucial for the following discussions.

Lemma 2. Suppose that c is such that $(cA + B)$ is invertible. Then the matrices $(cA + B)^{-1}A$ and $(cA + B)^{-1}B$ commute with each other.

Theorem 8. The equation $Ax' + Bx = 0$ has unique solutions for consistent initial conditions if and only if there exists a scalar c such that the matrix $(cA + B)$is invertible.

It is important to note that if A and B are $n \times n$ matrices, then the determinant $(\lambda A + B)$ is a polynomial of degree at most n. Therefore, either $(cA + B)$ is invertible for all but a finite number of values of c, or $(cA + B)$ is never invertible. Finding a value of C such that$(cA + B)$is invertible involves identifying a number that is not a root of a specific polynomial. This is generally considered simpler than finding the roots themselves. To simplify the formulas in the remainder of this section, we will introduce the following notation. Let

$$\hat{A}_C = (cA + B)^{-1}A, \hat{B}_C = (cA + B)^{-1}B, \hat{f}_C = (cA + B)^{-1}f \ (13)$$

where A and B are $n \times n$ matrices, f is a vector-valued function, and cc is such that $(cA + B)$ is invertible. If the matrices $\hat{A}_C, \hat{B}_C$, and $\hat{f}_C$ are used in a formula and the result is independent of the choice of c, we will omit the subscript.

Using Theorems 5, 6, 7, 8, and Lemma 2, we can now derive our most significant results.

Theorem 9. Suppose that the differential equation $Ax' + Bx = 0$ has unique solutions for consistent initial conditions. Let c be a number such that $(cA + B)$ is invertible. Define $\hat{A}, \hat{B}\ and\ \hat{f}$ according to equation (13). Assume $Ind(\hat{A}) = k$ Then the non-homogeneous equation $Ax' + Bx = f$ with initial condition $x(0) = x_0$ has a solution if and only if x_0 is of the form:

$$x_0 = \hat{A}\hat{A}^D q + \left(I - \hat{A}\hat{A}^D\right)\sum_{n=0}^{k-1}(-1)^n\left(\hat{A}\hat{B}^D\right)^n\hat{B}^D\hat{f}^{(n)}(0) \ (14)$$

for some vector q. A particular solution to $Ax' + Bx = f$ is given by

$$x = \hat{A}^D e^{-\hat{A}^D\hat{B}t}\int_a^t e^{\hat{A}^D\hat{B}s}\hat{f}(s)ds + (I - \hat{A}\hat{A}^D)\sum_{n=0}^{k-1}(-1)^n(\hat{A}\hat{B}^D)^n\hat{B}^D\hat{f}^{(n)} \ (15)$$

where a is arbitrary. The general solution of $Ax' + Bx = f$ is

$$x = e^{-\hat{A}^D\hat{B}t}\hat{A}\hat{A}^D q$$
$$+ \hat{A}^D e^{-\hat{A}^D\hat{B}t}\int_a^t e^{\hat{A}^D\hat{B}s}\hat{f}(s)ds + (I$$
$$- \hat{A}\hat{A}^D)\sum_{n=0}^{k-1}(-1)^n(\hat{A}\hat{B}^D)^n\hat{B}^D\hat{f}^{(n)} \quad (16)$$

The solution that satisfies $x(0) = x_0$ is obtained by setting $q = x_0$ and $a = 0$ in equation (16).

It is important to note that equation (7) has unique solutions for consistent initial conditions if and only if it has unique analytic solutions for consistent initial conditions. Additionally, the expressions given in equations (14), (15), and (16) are independent of the parameter c. This fact is established by the next theorem.

Theorem 10. Suppose that $A \; and \; B$ are n x n matrices and $(cA + B)^{-1}$ exists for some c. Then the matrices $\hat{A}_c^D\hat{A}_c, \hat{A}_c^D\hat{B}_c, \hat{A}_c^D(cA + B)^{-1}$, $\hat{B}_c^D(cA + B)^{-1}$, $\hat{A}_c^D\hat{B}_c$and $Ind(\widehat{A_c})$ are independent of c.

Note that $\hat{A}_c^D$ is not typically independent of c. However, if there exists a c such that $(cA + B)$ is invertible, then:

$$\lim_{c\to\infty}\frac{\hat{A}_c^D}{c} = \hat{A}\hat{A}^D \text{ and } \lim_{c\to\infty}\hat{A}_c^D = \hat{B}\hat{A}^D \quad (20)$$

Whether (17) can be used to derive a formula for$\hat{A}\hat{A}^D$and $\hat{B}\hat{A}^D$ for that is independent of c remains unknown.

In conclusion, we note that $N(A)\bigcap N(B) = \{0\}$ is not sufficient to guarantee that $(cA + B)$ is invertible for some c.

Example 4: Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

In this case $N(A)\bigcap N(B) = \{0\}$. but det $(cA + B) = 0$ for all values of c.

If there exists a value of$c$ such that $(cA + B)$ is invertible, then the matrix-valued function $(\lambda A + B)$ is referred to as a regular pencil. canonical form for regular pencils may be found[3]. Lemma 2 is a preliminary step in the development of such canonical forms.[7].

Greville in[8], demonstrated that the Drazin inverse of an $n \times n$ matrix A can be expressed as a polynomial in A, by the Cayley-Hamilton theorem, this polynomial can be assumed to have a degree of n or less. Greville also provided a sequential algorithm for computing the Drazin inverse in[9]. Additionally, Robert, in [10] developed a method specifically for calculating the Drazin inverse of matrices with index one, a case also discussed in [9].

In this section, we shall calculate the Drazin inverse if the eigenvalues of the matrix A are known.

Suppose that 0 is an eigenvalue of A with multiplicity$l$, and the distinct nonzero eigenvalues of A are λ_i with multiplicity $n_i \, for \, i = 1,2,\dots,r$. if $m = n_1 + n_2 + \cdots + n_r$ then $m + l = n$. Consider the following polynomial of degree $n - 1$:

$$p(\lambda)= \lambda^l(\alpha_0 + \alpha_1\lambda + \cdots + \alpha_{m-1}\lambda^{m-1}) \text{ (18)}$$

The determine the coefficients of $p(\lambda)$, solve the following m equations for the coefficients of $p(\lambda)$:

For $i = 1,2,3, \dots, r$

$$\frac{1}{\lambda_i} = p(\lambda_i)$$

$$\frac{-1}{\lambda_i{}^2} = p'(\lambda_i) \text{ (19)}$$

$$\vdots$$

$$\frac{(-1)^{n_i-1}(n_i-1)!}{(\lambda_i)^{n_i}} = p^{(n_i-1)}(\lambda_i)$$

It is straightforward to demonstrate that these equations (19) have a unique solution. Therefore, we obtain the following result.

Theorem 11. if $p(\lambda)$is defined by (18) and satisfies the conditions given in equation (19), then $p(A) = A^D$

Example 5. Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{bmatrix}$$

The eigenvalues of A are $\{0, 0, 1, \}$. According to Theorem 11, the Drazin inverse (A^D) given by

$$A^D = A^2(\alpha_0 I + \alpha_1 A)$$

where α_0 and $\alpha_1$are solutions to the following system of equations:

$$1 = \alpha_0 + \alpha_1$$

$$-1 = 2\alpha_0 + 3\alpha_1$$

Solving these equations yields$\alpha_0 = 4, \alpha_1 = -3$ Thus, we have:

$$A^D = A^2(\alpha_0 I + \alpha_1 A) = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}$$

It is well understood that computing the eigenvalues of a matrix can be challenging. The method proposed for computing the Drazin inverse A^D is effective for manual calculations and theoretical analysis. However, for large-scale problems, it is more practical to use methods that do not rely on accurately determining all eigenvalues. As discussed in [10], if the Jordan form of A is known, then A^Dcan be computed directly from this form. There are established methods for numerically computing the Jordan normal form (and consequently the eigenvalues and/or the core-nilpotent decomposition). For further information, see, [11].

Example 6. Consider the homogeneous differential equation

$$Ax' + Bx = 0 \ (20)$$

where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 2 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -27 & -22 & -17 \\ 18 & 14 & 10 \end{bmatrix} \ (21)$$

Note that A and B are singular matrices and do not commute. However, $(A + B)$ is invertible. To simplify the problem, we choose c and multiply the differential equation by$(A + B)^{-1}$ on the left to obtain:

$$\hat{A}x' + \hat{B}x = 0$$

where

$$\hat{A} = \frac{1}{3}\begin{bmatrix} -3 & -5 & -4 \\ 6 & 5 & -2 \\ -3 & 2 & 10 \end{bmatrix}, \hat{B} = \frac{1}{3}\begin{bmatrix} 6 & 5 & 4 \\ -6 & 2 & 2 \\ 3 & -2 & 7 \end{bmatrix} \ (22)$$

A unique solution can be guaranteed if and only if the initial vector $x(0)$)meets the following condition:

$$(\text{I-}\hat{A}A\hat{}^{D})x(0) = 0 \ (23)$$

The eigenvalues of$\hat{A}$ and $\hat{B}$ are $\{0, 1, 3\}$ and $\{0, 1, -2\}$, respectively. Using the method described in the previous section, the Drazin inverses $\hat{A}^D$and $\hat{B}^D$ can be computed as follows:

$$\hat{A}^D = \frac{1}{27}\begin{bmatrix} -27 & -41 & -28 \\ 54 & 77 & 46 \\ -27 & -34 & -14 \end{bmatrix}, \hat{B}^D = \frac{1}{12}\begin{bmatrix} 24 & 19 & 14 \\ -24 & -16 & -8 \\ 12 & 5 & -2 \end{bmatrix}$$

The calculation for Equation (23) can now be performed as follows:

$$9x_1(0) + 7x_2(0) + 5x_3(0) = 5 \ (24)$$

Given that the eigenvalues of $-\hat{A}^D\hat{B}$ are found to be $0, 0, \frac{2}{3}$, computing the matrix exponential is straightforward. The resulting matrix exponential is:

$$x(t) = e^{-\hat{A}^D\hat{B}t}x(0) = \frac{1}{18}\begin{bmatrix} 18 & 1 - e^{\frac{2}{3}t} & 2(1 - e^{\frac{2}{3}t}) \\ 0 & 26 - 8e^{\frac{2}{3}t} & 16(1 - e^{\frac{2}{3}t}) \\ 0 & 13(e^{\frac{2}{3}t} - 1) & 26(e^{\frac{2}{3}t} - 1) \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

Where $x_1(0), x_2(0), x_3(0)$satisfy (24).

Now consider the nonhomogeneous equation

$$Ax' + Bx' = b$$

where A, B are given by (21) and b is the constant vector $b = [1,2,0]^T$. Again we multiply on the left by $(A + B)^{-1}$to obtain $\hat{A}x' + \hat{B}x' = \hat{b}$ where A, B are given in (22) and:

$$\hat{b} = (A + B)^{-1}b = \frac{1}{9}\begin{bmatrix} -11 \\ 20 \\ -10 \end{bmatrix}$$

If the initial vector meets the requirement for the uniqueness (or existence) of solutions, then the solution can be determined using Theorem 9. In this particular scenario, since $\hat{A}$ has an index of 1, we have:

$$x(t) = e^{-\hat{A}^D\hat{B}t}\hat{A}\hat{A}^D q + \hat{A}^D e^{-\hat{A}^D\hat{B}t}\int_a^t e^{\hat{A}^D\hat{B}s}\hat{b}ds + \left(I - \hat{A}\hat{A}^D\right)\hat{B}^D\hat{b} \ (25)$$

Setting $t = 0$ gives

$$\left(I - \hat{A}\hat{A}^D\right)\left(x(0) - \hat{B}^D\hat{b}\right) = 0 \ (26)$$

As a necessary and sufficient condition for the existence of a solution with the initial value $x(0)$. one computer Equation (26) to be:

$$9x_1(0) + 7x_2(0) + 5x_3 + 1 = 0 \ (27)$$

Since$\hat{b}$ is a constant, the integral in Equation (25) can be evaluated using Formula (5). Given that $\hat{A}^D\hat{B}, \hat{A}\hat{B}$ all have an index of 1, Formula (5) simplifies to:

$$\int_a^t e^{\hat{A}^D\hat{B}s}\hat{b}ds = \left\{\hat{A}\hat{B}^D\left(e^{\hat{A}^D\hat{B}t} - I\right) + \left(I - \hat{A}\hat{A}^D\hat{B}\hat{B}^D\right)t\right\}\hat{b} \ (28)$$

By substituting Equation (28) into Equation (25) and simplifying, we obtain the final solution as:

$$x(t) = e^{-\hat{A}^D\hat{B}t}\hat{b}\left(x(0) - \hat{B}^D\hat{b}\right) + \hat{B}^D\hat{b} + \hat{A}^D\left(I - \hat{B}^D\hat{B}\right)t\hat{b} \ (29)$$

Evaluating Equation (29) yields:

$$x_1(t) = -\frac{1}{18}e^{\frac{2}{3}t}\left(x_2(0) + 2x_3(0)\right) - \frac{13}{18}x_2(0) - \frac{4}{9}x_3(0) - \frac{2}{9} - t$$

$$x_2(t) = -\frac{4}{9}e^{\frac{2}{3}t}\left(x_2(0) + 2x_3(0)\right) - \frac{13}{9}x_2(0) + \frac{8}{9}x_3(0) + \frac{2}{9} + 2t$$

$$x_3(t) = \frac{13}{18}e^{\frac{2}{3}t}\left(x_2(0) + 2x_3(0)\right) - \frac{13}{18}x_2(0) - \frac{4}{3}x_3(0) - \frac{10}{9} - t$$

where $x_1(0)$has been eliminated using Equation (27).

Conclusion

The research presented in this paper affirms the significance of the Drazin inverse as an essential mathematical tool for addressing the complexities associated with first-order linear singular differential equations. Utilizing the Drazin inverse, alongside the exponential matrix method, offers a robust framework for solving equations that are intractable through conventional means due to the singularity of the coefficient matrix. This work not only consolidates an understanding of the Drazin inverse's application in differential equations but also extends the frontier of how singular systems can be approached, potentially influencing future research directions in the field of applied mathematics and engineering.

References

1. C. Bu, K. Zhang, and J. Zhao, “Representations of the Drazin inverse on the solution of a class singular differential equations,” Linear Multilinear Algebr., vol. 59, no. 8, pp. 863–877, Aug. 2011, doi: 10.1080/03081087.2010.512291.

2. A. Ben and T. Greville, “Generalized inverses: theory and applications,” 2003, Accessed: Jul. 31, 2024. [Online]. Available: https://www.sidalc.net/search/Record/cat-unco-ar-60149/Description

3. S. L. Campbell, C. D. Meyer, and N. J. Rose, “APPLICATIONS OF THE DRAZIN INVERSE TO LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH SINGULAR CONSTANT COEFFICIENTS.,” SIAM J. Appl. Math., vol. 31, no. 3, pp. 411–425, 1976, doi: 10.1137/0131035.

4. X. Chen, C. S. Solutions, and X. Chen, “Solution of Time-Invariant State-Space Equations 1.1 Continuous-Time State-Space Solutions,” pp. 1–15, 2023.

5. “حل معادلات دیفرانسیل با ماتریس نمایی — به زبان ساده – فرادرس - مجله.” Accessed: Aug. 01, 2024. [Online]. Available: https://blog.faradars.org/حل-معادلات-دیفرانسیل/

6. “campbell1976 (1).pdf.”

7. M. M.-S. Review and undefined 1964, “Applications of the theory of matrices (fb grantmaker),” search.proquest.com, Accessed: Aug. 01, 2024. [Online]. Available: https://search.proquest.com/openview/ba0db0d4bad05a01b714d7fd5cab361e/1?pq-origsite=gscholar&cbl=30748

8. T.’N. E. GREVlLLE, Spectral generalized inverses... - Google Scholar.” Accessed: Aug. 01, 2024. [Online]. Available: https://scholar.google.com/scholar?hl=en&as_sdt=0%2C5&q=+T.’N.+E.+GREVlLLE%2C+Spectral+generalized+inverses+ofsquare+matrices%2C+M.R.C.+Tech.+Sum.+Rep.+823%2C+Mathematics+Research+Center%2C+University+of+Wisconsin%2C+Madison%2C+1967&btnG=

9. T. G.-L. A. and its Applications and undefined 1973, “The Souriau-Frame algorithm and the Drazin pseudoinverse,” ElsevierTNE GrevilleLinear Algebr. its Appl. 1973•Elsevier, Accessed: Aug. 01, 2024. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0024379573900219

10. S. Campbell, C. M. J.-L. A. and its Applications, and undefined 1975, “Continuity properties of the Drazin pseudoinverse,” ElsevierSL Campbell, CD Meyer JrLinear Algebr. its Appl. 1975•Elsevier, Accessed: Aug. 01, 2024. [Online]. Available: https://www.sciencedirect.com/science/article/pii/002437957590097X

11. V. K.-U. C. M. and Mathematical and undefined 1966, “On a method of solving the complete eigenvalue problem for a degenerate matrix,” Elsevier, Accessed:

Aug. 01, 2024. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0041555366900012