Vaqt bo‘yicha kasr tartibli uzulishli koeffitsiyentli diffuziya tenglamasi uchun aralash masala

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Annotatsiya: Bu maqolada diffuziya koeffitsiyenti uzulishga ega bo‘lgan kasr tartibli diffuziya tenglamasi uchun aralash masala qaralgan.

Kalit so‘zlar: kasr tartibli diffuziya tenglamasi, anomal diffuziya, 1-tur uzulish, Kaputo hosilasi

A mixed problem for the diffusion equation with a fractional-order discontinuity coefficient in time

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Abstract: In this article, mixed problem for the time-fractional diffusion equation with discontinuous diffusion coefficient is studied.

Keywords: fractional diffusion equation, anomalous diffusion, discontinuity, Caputo derivative

So‘nggi yillarda klassik diffuziyadan farqli bo‘lgan anomal diffuziya hodisalar ko‘p kuzatilmoqda.Shu jumladan, uzulishga ega bo‘lgan holat. Diffuziya koeffitsiyenti uzulishga ega bo‘lgan diffuziya tenglamasi Hald[6], Suzuki va Murayama[7], Pierce[8] maqolalarida uchraydi. Kasr tartibli koeffitsiyentlari uzluksiz bo‘lgan anomal diffuziya tenglamasi uchun teskari masala Cheng[1] maqolasida batafsil ko‘ril chiqilgan. Ushbu maqolada diffuziya koeffitsiyenti ma’lum bir nuqtada uzulishga ega bo‘lgan kasr tartibli anomal diffuziya tenglamasi uchun aralash masalani ko‘rib chiqamiz.

Biz quyidagi kasr tartibli differensial tenglamani ko‘rib chiqamiz:

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}(x,t)\right), \quad 0 < x < l, \quad 0 < t < T \tag{1}$$

$$u(x,0) = \delta(x) \tag{2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0, \quad 0 < t \le T \tag{3}$$

Bu yerda $T > 0, l > 0$ tayinlangan , $\delta(x)$ Dirakning delta funksiyasi va $\partial_t^{\alpha} u(x,t)$ - $u(x,t)$ funksiyaning Kaputo ma'nosidagi kasr tartibli hosilasi:

$$\partial_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} ds \qquad (4)$$

Bizda $0 < \alpha < 1$ va $p(x)$ funksiya x_0 nuqta uzulishga ega, ya'ni

$$p(x) = \begin{cases} a^2, & 0 < x < x_0, \\ b^2, & x_0 < x < l, \qquad a \neq b \end{cases} \qquad (5)$$

Biz (2) boshlang'ich shart Dirakning delta funksiyasi bo'lganligi sababli bu masala uchun kuchsiz yechim qaraymiz.Buning uchun biz (1)-(3) ga kuchsiz yechim uchun tegishli ta'rifni kiritishimiz va yechimni kuchsiz yechim ekanligini tekshirishimiz kerak. Kuchsiz yechim ta'rifi kiritilishi uchun zarur bo'lgan funksional fazolarni kiritamiz.

Birinchi biz $L^2(0,l)$ da A_p operatorni

$$A_p \psi(x) = \begin{cases} a^2 \psi''(x), & 0 < x < x_0 \\ b^2 \psi''(x), & x_0 < x < l \end{cases} \qquad (6)$$

ko'rinishida va uning aniqlanish sohasi $D(A_p)$ ni quyidagicha

$$D(A_p) = \left\{ \begin{array}{l|l} \psi \in C[0,l], & \psi(x_0 - 0) = \psi(x_0 + 0) \\ \psi \in C^1[0,x_0], \psi \in C^1[x_0,l] & a^2 \psi'(x_0 - 0) = b^2 \psi'(x_0 + 0) \\ \psi \in C^2(0,x_0), \psi \in C^2(x_0,l) & \psi'(0) = \psi'(l) = 0 \end{array} \right\}$$

kiritamiz.

Bundan ma'lumki, A_p operatorning $\lambda_n, n \in \mathbb{N}$ xos sonlari haqiqiy va oddiy sonlar bo'ladi va quyidagi shartni

$$0 = \lambda_1 < \lambda_2 < \dots, \quad \lim_{n \to \infty} \lambda_n = \infty. \qquad (7)$$

qanoatlantiradi.

A_p operatorning aniqlash sohasi $L_2(0,l)$ fazoning qism fazosini bo'ladi, ya'ni $D(A_p) \subset L_2(0,l)$.

Lemma 1. A_p operator uchun quyidagilar o'rinli:

1) $(A_p y, z) = (y, A_p z), \qquad \forall y, z \in D(A_p)$

2) $(A_p y, y) \geq C \|y\|^2$,

Isboti.

1. $L_2(0,l)$ da skalyar ko'paytmani qo'llaymiz.

$$(A_p y, z) = \int_0^l A_p yz dx = -\int_0^{x_0} a^2 y''(x) z(x) dx - \int_{x_0}^l b^2 y''(x) z(x) dx =$$

$$= -a^2 y'(x) z(x)\Big|_0^{x_0} + \int_0^{x_0} a^2 y'(x) z'(x) dx - b^2 y'(x) z(x)\Big|_{x_0}^l + \int_{x_0}^l b^2 y'(x) z'(x) dx =$$

$$= -a^2 y'(x_0 - 0) z(x_0 - 0) + a^2 y'(0) z(0) + a^2 y(x) z'(x)\Big|_0^{x_0} - \int_0^{x_0} a^2 y(x) z''(x) dx -$$

$$-b^2 y'(x_0 + 0) z(x_0 + 0) + b^2 y'(l) z(l) + b^2 y(x) z'(x)\Big|_{x_0}^l - \int_{x_0}^l b^2 y(x) z''(x) dx$$

Endi $D(A_p)$ operatorning aniqlash sohasidagi x_0 nuqtadagi $y'(0) = z'(l) = 0$ shartdan foydalansak,

$$(A_p y, z) = -a^2 y'(x_0 - 0) z(x_0 - 0) + a^2 y(x_0 - 0) z'(x_0 - 0) + b^2 y'(x_0 + 0) z(x_0 + 0) -$$

$$-b^2 y(x_0 + 0) z'(x_0 + 0) - \int_0^{x_0} a^2 y(x) z''(z) dx - \int_{x_0}^l b^2 y(x) z''(z) dx =$$

$$= a^2 \left[y(x_0 - 0) y'(x_0 - 0) - y'(x_0 - 0) y(x_0 - 0) \right] +$$

$$+ b^2 \left[y(x_0 + 0) y'(x_0 + 0) - y'(x_0 + 0) y(x_0 + 0) \right] + (y, A_p z)$$

hosil bo‘ladi va x_0 nuqtadagi ulash shartlani foydalanib,

$$(A_p y, z) = (y, A_p z)$$

ega bo‘lamiz.

2. $L_2(0, l)$ da skalyar ko‘paytmani qo‘llaymiz.

$$(A_p y, y) = \int_0^l A_p y \cdot y dx = -\int_0^{x_0} a^2 y''(x) y(x) dx - \int_{x_0}^l b^2 y''(x) y(x) dx =$$

$$= -a^2 y'(x) y(x)\Big|_0^{x_0} + \int_0^{x_0} a^2 y'(x) y'(x) dx - b^2 y'(x) y(x)\Big|_{x_0}^l + \int_{x_0}^l b^2 y'(x) y'(x) dx =$$

$$= -a^2 y'(x_0 - 0) y(x_0 - 0) + 0 - 0 + b^2 y'(x_0 + 0) y(x_0 + 0) + a^2 \int_0^{x_0} y'^2(x) dx + b^2 \int_{x_0}^l y'^2(x) dx =$$

$$= a^2 \int_0^{x_0} y'^2(x) dx + b^2 \int_{x_0}^l y'^2(x) dx$$

Endi $\frac{1}{2}\min\{a^2, b^2\}$ tanlaymiz , quyidagi o‘rinli bo‘ladi:

$$(A_p y, y) = a^2 \int_0^{x_0} y'^2(x) dx + b^2 \int_{x_0}^l y'^2(x) dx \geq \frac{1}{2}\min\{a^2, b^2\} \int_0^l y'^2(x) dx = \frac{1}{2}\min\{a^2, b^2\} \|y'\|^2_{L_2(0,l)}$$

Quyidagi baholashlarni bajaramiz:

$$y^2(x) = 2\int_0^x y(x) y'(x) dx \leq 2\left(\int_0^x y'^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^x y^2(x) dx \right)^{\frac{1}{2}} \leq 2\left(\int_0^l y'^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^x y^2(x) dx \right)^{\frac{1}{2}}$$

$$\int_0^l y^2(x)dx \le 2\left(\int_0^l y'^2(x)dx\right)^{\frac{1}{2}} \int_0^l \left(\int_0^x y^2(x)dx\right)^{\frac{1}{2}} dx \le 2\|y'\|_{L_2(0,l)} \|y\|_{L_2(0,l)}\, l$$

$$\|y\|^2_{L_2(0,l)} \le 2l\|y'\|_{L_2(0,l)} \|y\|_{L_2(0,l)}$$

$$\frac{1}{4l^2}\int_0^l y^2(x)dx \le \int_0^l y'^2(x)dx$$

Bundan quyidagi baholash kelib chiqadi:

$$(A_p y, y) \ge \frac{\min\{a^2, b^2\}}{2}\int_0^l y'^2(x)dx \ge \frac{\min\{a^2, b^2\}}{8l^2}\int_0^l y^2(x)dx = C\|y\|^2_{L_2(0,l)}$$

Ya’ni shunday $C = \frac{\min\{a^2, b^2\}}{8l^2}$ mavjudki, $L_2(0,l)$ da quyidagi tengsizlik o‘rinli:

$$(A_p y, y) \ge C\|y\|^2.$$

Lemma isbotlandi.

Biz A_p operatorning spektri sof diskret bo‘lgani va faqat xos qiymatlaridan tuzilgan o‘z-o‘ziga qo‘shma kengaytirib, bu kengaytmani $\hat{A}_p$ bilan belgilaymiz. $A_p = \hat{A}_p$

Lemma 2. A_p operator x_0 nuqtadagi ulash shartlar va

$$\frac{(l - x_0)a}{x_0 b} = \frac{n}{k} \in \mathbb{Q}$$

Shart orqali quyidagi xos sonlarga ega bo‘ladi:

$$\lambda_n = \frac{a^2}{4x_0^2}\pi^2 n^2 = \frac{b^2}{4(l - x_0)^2}\pi^2 k^2 \qquad (8)$$

λ_n xos songa mos xos funksiyani φ_n bilan belgilaymiz va u $\varphi(0) = 1$ ni qanoatlantiradi.

Bizda $\varphi_n(x)$ xos funksiya

$$\varphi_n(x) = \begin{cases} \cos\frac{\sqrt{\lambda_n}}{b}(l - x_0)\cos\frac{\sqrt{\lambda_n}}{a}x, & 0 < x < x_0 \\ \cos\frac{\sqrt{\lambda_n}}{a}x_0\cos\frac{\sqrt{\lambda_n}}{b}(l - x), & x_0 < x < l \end{cases}$$

ko‘rinishda bo‘ladi. φ_n xos funksiyalar $L^2(0,\ell)$ da to‘la orthogonal bo‘ladi. $(\cdot,\cdot)$ - $L^2(0,\ell)$ dagi skalyar ko‘paytmani aniqlaydi va uning ko‘rinishi quyidagicha

$$(f, g) = \int_0^l f(x)g(x)dx$$

bo‘ladi. Bu skalyar ko‘paytma orqali $L^2(0,\ell)$ da normani kiritamiz

$$\|\varphi\|_{L^2(0,\ell)} = \|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}}.$$

Quyidagi belgilashni kiritamiz: $\rho_n = \|\varphi_n\|^{-2}$. U holda xos funksiyalar kengaytmasi

$$\psi = \sum_{n=1}^{\infty} \rho_n (\psi, \varphi_n)\varphi_n.$$

Bundan tashqari, $\rho_n, n \in \mathbb{N}$ asimptotik xarakteriga ega.

Lemma 3. Shunday $c_0 > 0$ mavjudki,

$$\rho_n = c_0 + o(1)$$

Endi biz ixtiyoriy o‘zgarmas $M > 0$ ni tanlab $L^2(0,l)$ da $A_{p,M}$ operatorni

$$\begin{cases} (A_{p,M}\psi)(x) = -\dfrac{d}{dx}\left(p(x)\dfrac{d}{dx}\psi(x)\right) + M\psi, \quad 0 < x < \ell, \\ \mathrm{D}(A_{p,M}) = \left\{\psi \in H^2(0,\ell); \dfrac{d\psi}{dx}(0) = \dfrac{d\psi}{dx}(\ell) = 0\right\}. \end{cases}$$

aniqlaymiz. Bu operatorning xos sonlarning to‘plami $\lambda_n^{(M)} = \lambda_n + M_n \in \mathbb{N}$ bo‘ladi va $\lambda_n^{(M)} > 0, n \in \mathbb{N}$. Endi $\kappa > 0$ uchun quyidagi $\mathrm{D}\left(A_{p,M}^{\kappa}\right)$ funksional fazoni

$$\mathrm{D}\left(A_{p,M}^{\kappa}\right) = \left\{\psi \in L^2(0,\ell); \sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi,\varphi_n)|^2 < \infty\right\}$$

kiritamiz. Bu $\mathrm{D}\left(A_{p,M}^{\kappa}\right)$ fazo quyidagi norma bilan Banax fazosi bo‘ladi:

$$\|\psi\|_{\mathrm{D}\left(A_{p,M}^{\kappa}\right)} = \left\{\sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi,\varphi_n)|^2\right\}^{\frac{1}{2}}.$$

Biz agar $0 \le \kappa < \frac{3}{4}$ bo‘lsa, $\mathrm{D}\left(A_{p,M}^{\kappa}\right) = H^{2\kappa}(0,l)$ ga ega bo‘lamiz. $\mathrm{D}\left(A_{p,M}^{\kappa}\right) \subset L^2(0,\ell)$ ligidan, o‘z-o‘ziga qo‘shma $L^{2,p}(0,l)'$ ni aniqlashdan, $\mathrm{D}\left(A_{p,M}^{\kappa}\right) \subset L^{2,p}(0,l) \subset \left(\mathrm{D}\left(A_{p,M}^{\kappa}\right)\right)'$ ga ega bo‘lamiz. Biz $\mathrm{D}\left(A_{p,M}^{-\kappa}\right) = \left(\mathrm{D}\left(A_{p,M}^{\kappa}\right)\right)'$ deb belgilash kiritamiz. $f \in \left(\mathrm{D}\left(A_{p,M}^{-\kappa}\right)\right)$ va $\psi \in \left(\mathrm{D}\left(A_{p,M}^{\kappa}\right)\right)$ uchun f ning ψ ga ta’sirini ${}_{-k}< f,\psi >_k$ bilan belgilaymiz. $\mathrm{D}\left(A_{p,M}^{-\kappa}\right)$ fazo $\|f\|_{\mathrm{D}\left(A_{p,M}^{-\kappa}\right)} = \left\{\sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{-2\kappa} \|^2\right\}^{\frac{1}{2}}$ norma bilan Banax fazosi bo‘ladi. Endi $0 < \tau < \frac{1}{2}$ ni tayinlaymiz. Sobolevning joylashtirish teoremasiga ko‘ra, $\delta \in D\left(A_{p,M}^{-\frac{1}{4}-\tau}\right)$ va $D\left(A_{p,M}^{-\frac{1}{4}-\tau}\right)$ da $\delta = \sum_{n=1}^{\infty} \rho_n \varphi_n$ ga ega bo‘lamiz. Biz $<\cdot,\cdot> = {}_{-\frac{1}{4}-\varepsilon}<\cdot,\cdot>_{\frac{1}{4}+\varepsilon}$ belgilash kiritib olamiz. Brezis[4] ga ko‘ra agar $f \in L^2(0,l)$ va $\psi \in D\left(A_{p,M}^{\frac{1}{4}+\varepsilon}\right)$ bo;lsa, u holda $< f,\psi > = (f,\psi)$ bo‘rinli.

Ta’rif. Agar u funksiya quyidagi shartlarni bajarsa, (2.1.1)-(2.1.3) masalaning kuchsiz yechimi deyiladi:

$$\begin{cases} u(\cdot,t) \in L^2(0,\ell), \qquad 0 < t \le T, \\ u \in C\left([0,T]; D\left(A_{p,M}^{-\frac{1}{4}-\tau}\right)\right), \\ \dfrac{\partial}{\partial t}u, \partial_t^{\alpha}u, A_{p,M}u \in C\left((0,T]; D(A_{p,M}^{-\frac{1}{4}-\tau})\right), \end{cases}$$

$$\lim_{t\to 0} \| u(\cdot,t) - \delta \|_{D\left(A_{p,M}^{-\frac{1}{4}-\tau}\right)} = 0$$

$$\langle \partial_t^{\alpha} u(\cdot,t), \psi \rangle + (u(\cdot,t), A_p\psi) = 0, \quad t \in (0,T], \psi \in D(A_p).$$

Teorema. $u(x,t) \in C\left([0,T]; D\left(A_{p,M}^{-\frac{1}{4}-\tau}\right)\right)$ funksiya (1)-(3) masalaning kuchsiz yechim bo‘ladi va uning ko‘rinishi quyidagicha bo‘ladi:

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\alpha_n t^{\alpha})\varphi_n(x).$$

Bu yerda $\alpha > 0$ va $\beta \in \square$ uchun $E_{\alpha,\beta}(z)$ Mittag-Leffler funksiyasidir.

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