Funksiyaning monotonlik xossasi yordamida tengsizliklarni isbotlash

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Annotatsiya: Maqolada funksiyaning monotonlik xossasi yordamida ba’zi tengsizliklarni isbotlash usullari keltirib o‘tilgan.

Kalit so‘zlar: funksiya, funksiyaning monoton va ekstrimumlik xossalari, tengsizlik, isbot, masala va yechim

Proving inequalities using the monotonicity property of a function

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Abstract: The article presents methods of proving some inequalities using the monotonicity property of a function.

Keywords: function, monotone and extremum properties of a function, inequality, proof, problem and solution

Ma’lumki, agar $f(x)$ funksiya $[a,b]$ kesmada aniqlangan monoton, yoki bu kesma ichida faqat bitta ekstremumga ega bo‘lib, bu ekstremum uning minimumi bo‘lsa, u holda bu funksiya o‘zining eng katta qiymatlarini kesmaning chegara nuqtalarida erishadi. Shu kabi, agar bu ekstremum uning maksimumi bo‘lsa, u holda bu funksiya o‘zining eng kichik qiymatlarini kesmaning chetki nuqtalarida erishadi. Funksiyaning bu xossasidan ba’zi tengsizliklarni isbotlashda foydalanish mumkin. Quyida funksiyaning monotonlik xossasi yordamida ayrim tengsizliklarni isbotlarini keltirib o‘tamiz.

1-masala. Agar x_1, x_2, x_3 haqiqiy sonlar bo‘lib, $0 \le x_i \le 1$, $i = 1,2,3$ shartni qanoatlantirsa $x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_3x_1 \le 1$ tengsizlikni isbotlang.

Yechim. $x_1 = x$ belgilash kiritib,

$$f(x) = x + x_2 + x_3 - x\ x_2 - x_2x_3 - x_3x = x(1 - x_2 - x_3) + x_2 + x_3 - x_2x_3\ ,\ x \in [0,1]$$

funksiyani hosil qilamiz. Bu funksiya x ga nisbatan chiziqli funksiya bo‘lgani uchun monoton bo‘lib, u o‘zining eng katta qiymatlariga $[0,1]$ kesmaning chetki nuqtalarida erishadi. Shuning uchiun

$$f(0) = x_2 + x_3 - x_2x_3 = 1 + (1 - x_3)(x_2 - 1) \le 1,$$
$$f(1) = 1 - x_2 - x_3 + x_2 + x_3 - x_2x_3 = 1 - x_2x_3 \le 1.$$

Demak $f(x) \le 1$, bundan esa isbotlanishi talab qilingan

$x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_3x_1 \le 1$ tengsizlik hosil bo‘ladi.

2-masala. Agar *a, b, c* uchburchak tomonlari bo‘lib, $a + b + c = 1$ bo‘lsa, u holda $a^2 + b^2 + c^2 + 4abc < \frac{1}{2}$ tengsizlikni isbotlang.

Yechim. a, b, c uchburchak tomoni bo‘lgani uchun $0 < a, b, c < \frac{1}{2}$ bo‘ladi $c = x;\ b = 1 - a - x$ belgilash olsak,

$$f(x) = a^2 + (1 - a - x)^2 + x^2 + 4ax(1 - a - x) - \frac{1}{2}$$

funksiyani qaraymiz. Natijada

$$f'(x) = -2(1 - a - x) + 2x - 4ax + 4a(1 - a - x) = -2 + 2a + 2x + 2x - 4ax + 4a - 4a^2 - 4ax =$$
$$= 6a - 2 + 4x(1 - 2a) - 4a^2 = -2(2a^2 - 3a + 1) + 4x(1 - 2a) = -2(a - 1)(2a - 1) + 4x(1 - 2a) =$$
$$= 4(1 - a)(a - \frac{1}{2}) - 8x(a - \frac{1}{2}) = const + 8(\frac{1}{2} - a)x$$

bo‘lib funksiya monoton o‘sadi.

U o‘zining eng katta qiymatini $x = \frac{1}{2}$ da erishadi.

$$f(x) < f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(1 - \frac{1}{2} - a\right)^2 + a^2 + 4a \cdot \frac{1}{2}\left(1 - a - \frac{1}{2}\right) - \frac{1}{2} =$$
$$= \frac{1}{4} + \frac{1}{4} - a + a^2 + a^2 + a - 2a^2 - \frac{1}{2} = 0$$

Demak, $a^2 + b^2 + c^2 + 4abc < \frac{1}{2}$ tengsizlik isbotlandi.

3-masala. $a^2 + b^2 + c^2 \le a^2b + b^2c + c^2a + 1$ tengsizlikni isbotlang, bunda $0 \le a, b, c \le 1$
.

Yechim. [0;1] kesmada $f(x) = x^2(1 - b) - c^2x + b^2 + c^2 - b^2c - 1$ funksiyani qaraymiz (bunda *a*=*x* deb qaradik).

$$f'(x) = 2(1 - b)x - c^2$$

Agar $b \ne 1$ bo‘lsa, u holda $f'(x)$ o‘sadi, demak $f(x)$ funksiya monoton xossaga ega va o‘zini eng katta qiymati [0;1] kesma chegarasida erishadi.

$$f(0) = b^2 + c^2 - b^2c - 1 = (1 - c)(b^2 - (1 + c)^2) \le 0$$
$$f(1) = 1 - b^2 - c^2 + b^2 + c^2 - b^2c - 1 = b(b - 1) - b^2c \le 0$$

Shuning uchun $f(x) \le 0 \quad x \in [0;1]$ va demak $f(0) \le 0.$

$b = 1$ bo‘lsa ham isbot shu kabi olib boriladi.

4-masala. Agar $x \in [-1,1]$ barcha x lar uchun $\left|ax^2 + bx + c\right| \le 1$ tengsizlik o‘rinli bo‘lsa, u holda $\left|cx^2 - bx + a\right| \le 2$ tengsizlikni isbotlang.

Yechim.

$$\left|cx^2 - bx + a\right| \le \left|cx^2 - c\right| + \left|cx^2 - bx + a\right| = \left|c\right|(1 - x^2) + \left|cx^2 - bx + a\right| \le \left|c\right| + \left|cx^2 - bx + a\right|, \quad x \in [0,1]$$

Agar $x = 0$ bo‘lsa, u holda $\left|ax^2 + bx + c\right| \le 1$ tengsizlidan $\left|c\right| \le 1$ kelib chiqadi.

Shuning uchun $\left|cx^2 - bx + a\right| \le 1 + \left|c - bx + a\right|$.

Endi $f(x) = \left|c - bx + a\right|$ funksiya o‘zining eng katta qiymatiga [-1;1] kesma chegarasida erishishida qoldi.

Demak, $f(x) \le \max(f(-1); f(1)) = \max(\left|c + b + a\right|; \left|c - b + a\right|) \le 1$. Bundan esa,

$$\left|cx^2 - bx + a\right| \le 1 + 1 = 2$$ kelib chiqadi.

5-masala. $0 \le a, b, c \le 1$ sonlari uchun

$$\frac{a}{b + c + 1} + \frac{b}{c + a + 1} + \frac{c}{a + b + 1} + (1 - a)(1 - b)(1 - c) \le 1$$

tengsizlikni isbotlang.

Yechim. [0;1] kesmada $\frac{x}{b + c + 1} + \frac{b}{c + x + 1} + \frac{c}{x + b + 1} + (1 - x)(1 - b)(1 - c)$ funksiyani qaraymiz. Uning hosilasi $d + \frac{b}{(c + x + 1)^2} + \frac{c}{(x + b + 1)^2}; \quad d = const$.

ko‘rinib turibdiki $f'(x)$ $[0;1]$ kesmada o‘suvchi funksiya va demak $f(x)$ monoton xossaga ega, u o‘zini eng katta qiymatini 0 yoki 1 nuqtada qabul qiladi.

Demak, $f(x) \le \max(f(0); f(1)) \quad a = 0, \ a = 1$ va xuddi shunday,

$$f(x) \le \max \quad b \ bo'lsa \quad b = 0, \quad b = 1$$

$$f(x) \le \max \quad c \ bo'lsa \quad c = 0, \quad c = 1$$

ekanligini ko‘ramiz. Quyidagi (0,0,0), (0,0,1), (0,1,1), (1,1,1) sonlar uchliklari uchun tenglik bajariladi.

Foydalanilgan adabiyotlar

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