Ayrim qiziqarli mulohazalarni hamda tengsizliklarni isbotlash

A.Ibragimov
O.Pulatov
Sh.Xolmurodov
O‘zbekiston-Finlandiya pedagogika instituti

Annotatsiya: Sonlar nazariyasining ayrim sodda xossalari yordamida ba’zi qiziqarli mulohaza, tenglik va tengsizliklarning isboti qaralgan.

Kalit so‘zlar: daraja, natural, tenglik va tengsizlik

Proof of some interesting statements and inequalities

A.Ibragimov
O.Pulatov
Sh.Kholmurodov
Uzbek-Finnish Pedagogical Institute

Abstract: The proof of some interesting statements, equalities and inequalities using some simple properties of number theory is considered.

Keywords: degree, natural, equality and inequality

1. Har qanday natural a uchun $a^3 - 1$ soni ikkilik daraja bo‘la olmasligini isbotlang.

Isbot: $a^3 - 1 = 2^k \Rightarrow (a-1)(a^2 + a + 1) = 2^k \Rightarrow a - 1 = 2^m$ va $a^2 + a + 1 = 2^{k-m}$, $K \geq 2m$, $K, m \in \mathbb{N} \Rightarrow a = 2^m + 1$

$\Rightarrow a^2 + a + 1 = (2^m + 1)^2 + 2^m + 2 = 2^{2m} + 2^{m+1} + 2^m + 3$

$\Rightarrow 2^{k-m} = 2^{2m} + 2^{m+1} + 3 \Rightarrow$ ziddiyat $\Rightarrow \Delta$

2. Hamma natural a sonlarini toping, shundayki $a^3 + 1$ - uchlik darajasi bo‘lsin.

Yechim: $a^3 + 1 = 3^k$, $K \in \mathbb{N}$

$\Rightarrow (a+1)(a^2 - a + 1) = 3^k \Rightarrow a + 1 = 3^m$ va $a^2 - a + 1 = 3^{k-m}$, $K \geq 2m$.

1-hol. $K = 2m$ bo‘lsa $\Rightarrow a^2 - a + 1 = a + 1 = 3^m$

$\Rightarrow a^2 - 2a = 0 \Rightarrow a = 0$ yoki $a = 2 \Rightarrow a = 2 \Rightarrow a^3 + 1 = 9 = 3^2 \Rightarrow a = 2$

2-hol. $K > 2m$ bo‘lsa, $a > 2$ $K \geq 2m + 1$ $a + 1 = 3^m$ $a = 3^m - 1$

$a^2 - a + 1 = (3^m - 1)^2 - 3^m + 2 = 3^{2m} 2 \cdot 3^m + 1 - 3^m + 2 = 3^{2m} - 3^{m+1} + 3$

$3^{2m} - 2 \cdot 3^m + 1 - 3^m + 2 = 3^{2m} - 3^{m+1} + 3$ $3^{2m} - 3^{m+1} + 3 = 3^{k-m}$

$3^{2m} + 3 = 3^{k-m} + 3^{m+1}$ $3(3^{2m-1} + 1) = 3^{m+1}(3^{k-2m-1} + 1)$

$3^{2m-1} + 1 = 3^m(3^{k-2m-1} + 1) \Rightarrow \emptyset$ Javob: $a = 2$ Δ

3. $2^a + 1 \vdots b$, $2^b + 1 \vdots c$, $2^c + 1 \vdots a$ ni qanoatlantiruvchi o‘zaro tub, har xil 1 dan kata bo‘lgan a, b, c natural sonlar mavjudmi?

Yechim: Ushbu shartlarni qanoatlantiruvchi a, b, c lar mavjud. $a = 3$ bo‘lsin.

$\Rightarrow 2^a + 1 = 9 \Rightarrow 9 \vdots b$, $b = 9$ bo‘lsin.$\Rightarrow 2^b + 1 = 513 \Rightarrow 513 \vdots c \Rightarrow 27 \cdot 19 \vdots c$ $\Rightarrow c = 19$ bo‘lsin.$\Rightarrow 2^c + 1 = 2^{19} + 1 \vdots 3 \Rightarrow 2^c + 1 \vdots a$, $(a, b, c) = (3,9,19)$ Δ

4. Berilgan N natural soni uchun barcha qo‘shni raqamlar yig‘indisini hisoblang (masalan, $N = 35207$ uchun yig‘indilar {8,7,2,7}. Ushbu yig‘indilar orasida barcha sonlar 1 dan 9 gacha bo‘ladigan eng kichik N ni toping.

Yechim: N - o‘n xonali bo‘ladi. Eng kichikni topish kerak. N ning 1-raqami 1 ga teng

$\Rightarrow$ N=1021324354▲ [1]

5. Quyidagi tenglamani natural sonlar to‘plamida yeching

EKUK(a;b)+EKUB(a;b)=a·b

Yechim: EKUB(a,b)=d bo‘lsin. $a = a_1 \cdot d$, $b = b_1 \cdot d$ bo‘lsin, bu yerda $(a_1, b_1) = 1$.

EKUK(a, b) = $a_1 \cdot b_1 \cdot d \Rightarrow a_1 \cdot b_1 \cdot d + d = a_1 \cdot b_1 \cdot d^2 \Rightarrow a_1 \cdot b_1 + 1 = a_1 \cdot b_1 \cdot d$

$\Rightarrow a_1 \cdot b_1(d-1) = 1 \Rightarrow a_1 \cdot b_1 = 1$ va $d = 2 \Rightarrow a_1 = b_1 = 1$

$\Rightarrow a = a_1 d = 2, b = b_1 d = 2 \Rightarrow (a, b) = (2,2)$, ▲

6. Quyidagi tenglikni qanoatlantiruvchi barcha natural sonlarni toping: $n! = \overline{12 \ldots n}$

Yechim: Lemma: $a, b \in \mathbb{N}$ uchun $a! \geq a \cdot b$ o‘rinli.

Isboti:

b-n xonali son bo‘lsin. $n \in \mathbb{N}$

$\Rightarrow \overline{ab} > a \cdot 10^n \Rightarrow 10^n > b$ ni isbotlash yetarli. $10^n > b$ - bu esa to‘g‘ri $\Rightarrow$ ▲ lemma isbotlandi. Lemma va induksiyadan foydalansak, $\overline{12 \ldots n} > n!$ Ga ega bo‘lamiz, bu yerda $n > 1$ da yechim mavjud emas $n = 1$. Javob: $n = 1$ ▲

7. Natural sonlar a va b 1 000 000 dan katta bo‘lib, shunday shartni qanoatlantirsin:

$(a+b)^3$ soni ab ga bo‘linadi. Isbotlangki, $|a - b| > 10000$.

Isbot: $(a + b)^3 \vdots ab \Rightarrow a^3 + b^3 + 3ab(a + b) \vdots ab \Rightarrow a^3 + b^3 \vdots ab$ (1)

$ab \vdots p$, $a^3 \vdots p^\alpha$, $a^3 \vdots p^{\alpha+1}$, $b^3 \vdots p^\beta$, $b^3 \vdots p^{\beta+1}$, $\alpha \geq \beta > 0$ (*)

bo‘lsa, bu yerda p-tub son va $\alpha, \beta \in N \Rightarrow ab \vdots p^{\alpha+\beta}$ va $ab \vdots p^{\alpha+\beta+1}$ (2)

(*) dan $a^3 + b^3 \vdots p^\beta$, $a^3 + b^3 \vdots p^\alpha$, ammo (2)

$\Rightarrow a^3 + b^3 \vdots p^{\alpha+\beta} \Rightarrow a^3 \vdots p^{\alpha+\beta}$ va $b^3 \vdots p^{\alpha+\beta}$

$\Rightarrow$ (1) dan $\Rightarrow a^3 \vdots ab$ va $b^3 \vdots ab \Rightarrow a^3 - b^3 - 3ab(a - b) \vdots ab$

$\Rightarrow (a - b)^3 \vdots ab \Rightarrow |a - b|^3 \geq ab \Rightarrow |a - b|^3 > 10^6 \cdot 10^6 = 10^{12}$

$\Rightarrow |a - b| > 10^4 \Rightarrow |a - b| > 10000$ ▲

[1] Ayniyat yordamida tengsizliklarni isbotlash, A Ibragimov, O Pulatov, B Sag‘dullayeva, A Ibrahimov… - Science and Education, 2023

8. N soni 81 ga bo‘linmasa ham, uni 3 ga bo‘linadigan uchta butun son kvadratlarining yig‘indisi shaklida tasvirlash mumkin. Isbotlangki, uni 3 ga bo‘linmaydigan uchta butun son kvadratlari yig‘indisi shaklida ham yozish mumkin.

Isbot: $N = (3a)^2 + (3b)^2 + (3c)^2$, bu yerda $a, b, c \in N$. a, b, c larning hammasi bir vaqtda 3 ga bo‘lina olmaydi, chunki $N \vdots 81 \Rightarrow a + b + c \vdots 3$.

$N = 9a^2 + 9b^2 + 9c^2 = (2a + 2b - c)^2 + (2b + 2c - a)^2 + (2c + 2a - b)^2$

Endi $2a + 2b - c \vdots 3$, $2b + 2c - a \vdots 3$, $2c + 2a - b \vdots 3$ larni isbotlash yetarli.

$2(a + b) - c = 2(a + b + c) - 3c \vdots 3$, chunki $a + b + c \vdots 3$.

Xuddi shunday $2b + 2c - a \vdots 3$ va $2c + 2a - b \vdots 3$

$N = 3a^2 + 3b^2 + 3c^2 = (2a + 2b - c)^2 + (2b + 2c - a)^2 + (2c + 2a - b)^2$

▲

9. Qanday natural n lar uchun musbat ratsional, lekin butun bo‘lmagan a va b sonlari topiladiki, $a + b$ va $a^n + b^n$ butun sonlar bo‘ladi?

Yechim: 1-hol. n-juft bo‘lsin. n=2k, k∈N. a va bi lar ratsional sonlar va yig‘indisi butun son. $a = \frac{p}{d}$ va $b = \frac{q}{d}$ ko‘rinishda bo‘ladi. $p + q \vdots d$ bu yerda $\frac{p}{d}$ va $\frac{q}{d}$ qisqarmas kasrlar.

$p^n + q^n = (p^{2k} - q^{2k}) + 2q^{2k} = (p^2 - q^2)(p^{2k-2} + p^{2k-4}q^2 + \cdots + q^{2k-2}) +$

$+2q^{2k} = (p + q) \cdot k + 2q^{2k}$, bu yerda

$K = (p - q)(p^{n-2} + p^{n-4}q^2 + \ldots + q^{n-2}$ - butun son.

$a = \frac{p}{d}$ va $b = \frac{q}{d}$ $a^n + b^n = \frac{p^n + q^n}{d^n}$ $p^n + q^n \vdots d^n$ $p^n + q^n \vdots d$ (1)

$p^n + q^n = (p + q) \cdot K + 2 \cdot q^{2k}$ $a + b$ - butun. $\Rightarrow p + q \vdots d$ (2)

(1) va (2) $\Rightarrow 2 \cdot q^{2k} \vdots d$. Bizga ma’lumki, $\frac{q}{d}$ - qisqarmas $\Rightarrow q \vdots d \Rightarrow 2 \vdots d \Rightarrow d = 2$.

$\frac{p}{d}$ va $\frac{q}{d}$ lar qisqarmas bo‘lganligi uchun p, q - toq.

n - juftligidan p^n va q^n lar toq sonlarning kvadratlari. Demak, ularning har biri 4 ga bo‘lganda 1 qoldiq beradi. $p^n + q^n$ ni 4 ga bo‘lganda 2 qoldiq beradi. Ammo $p^{2k} + q^{2k} \vdots d^{2k} \Rightarrow p^n + q^n \vdots 2^{2k}$ va $p^n + q^n \vdots 4 \Rightarrow$ ziddiyat $\Rightarrow$ n - juft bo‘ladigan a, b lar mavjud emas. ▲

2-hol. n - toq bo‘lsin. $a = \frac{1}{2}$, $b = \frac{2^n - 1}{2}$ ni tekshiramiz. [2]

$a + b = \frac{1}{2} + \frac{2^n}{2} - \frac{1}{2} = 2^{n-1}$ $a + b = 2^{n-1}$ - butun

$a^n + b^n = \frac{1}{2^n} + \frac{(2^n - 1)^n + 1}{2^n}$, n - toqligi uchun $(z^n - 1)^n + 1 \vdots 2^n$

Javob: n - toqlarda bajariladi. ▲

10. Agar natural son N uchta butun son kvadratlarining yig‘indisi ko‘rinishida ifodalanib, ushbu sonlarning har biri 3 ga bo‘linadigan bo‘lsa, u holda N uchta butun

[2] Tengsizliklarni yechishning noodatiy usuli, A Ibragimov, O Pulatov, I Teshaboyeva, A - Science and Education, 2023

son kvadratlarining yig‘indisi sifatida ifodalanib, bu sonlarning hech biri 3 ga bo‘linmaydigan ko‘rinishga ham ega bo‘lishini isbotlash talab qilinadi.

Isbot: 1-son $3^n \cdot a$ ga bo‘lsin, bunda n∈N $a \not\vdots 3$.

2- va 3-sonlar 3^n ga bo‘linsin. Shartdan N ni $9^n(a^2+b^2+c^2)$ (1)

Ko‘rinishda yozish mumkin, bu yerda n∈N a, b, c ∈ Z.

Lemma: (1) ko'rinishdagi istalgan sonni $9^n(x^2+y^2+z^2)$ ko'rinishida tasvirlash mumkin, bunda x, y, z ∈ Z va $z, y, z \not\vdots 3$.

Isboti: Umumiylikka zarar yetkazmasdan $a+b+c \not\vdots 3$ deb olishimiz mumkin.

$$9(a^2+b^2+c^2)=(4a^2+4b^2+c^2)+(4b^2+4c^2+a^2)+(4c^2+4a^2+b^2)=$$
$$=(2a+2b-c)^2+(2b+2c-a)^2+(2c+2a-b)^2$$
$$9(a^2+b^2+c^2)=(2a+2b-c)^2+(2b+2c-a)^2+(2c+2a-b)^2$$

$2a+2b-c=2(a+b+c)-3c$, bu yerda $2(a+b+c) \not\vdots 3$ va $3c \vdots 3$

$2a+2b-c \not\vdots 3$, xuddi shunday $2b+2c-a \not\vdots 3$, $2c+2a-b \not\vdots 3$

$$9(a^2+b^2+c^2)=9\cdot 9^{n-1}(a^2+b^2+c^2)=$$
$$=9^{n-1}((2a+2b-c)^2+(2b+2c-a)^2+(2c+2a-b)^2)$$

Lemma isbotlandi.

Lemmaga ko‘ra $9^n(a^2+b^2+c^2)$ ni $9^{n-1}(x^2+y^2+z^2)$ ko‘rinishida tasvirlash mumkin, bunda $x, y, z \not\vdots 3$. Xuddi shunday, lemmani n marta qo‘llasak, $9^n(a^2+b^2+c^2)$ ni $9^{1-1}(m^2+n^2+p^2)$ ko‘rinishida tasvirlash mumkin, bunda $m, n, p \not\vdots 3 \Rightarrow N = m^2+n^2+p^2, m, n, p \not\vdots 3 \Rightarrow$ ▲

11. Isbotlangki, shunday 4 ta butun son a,b,c,d mavjudki, ularning modul bo‘yicha qiymatlari 1 000 000 dan katta va quyidagi tenglikni qanoatlantiradi:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{abcd}$$

Isbot: Qandaydir n ∈ N va 1000000 sonni qaraymiz.

Demak, biz $(-n, n+1, n(n+1)+1, n(n+1)(n(n+1)+1)+1)$

sonlar to‘rtligi masala shartini qanoatlantirishini isbotlasak, yetarli bo‘ladi.

$$\frac{1}{-n}+\frac{1}{n+1}+\frac{1}{n(n+1)+1}+\frac{1}{n(n+1)(n(n+1)+1)+1}=$$
$$\frac{-1}{n(n+1)}+\frac{1}{n(n+1)+1}+\frac{1}{n(n+1)(n(n+1)+1)+1}=$$
$$\frac{1}{n(n+1)(n(n+1)+1)}+\frac{1}{n(n+1)(n(n+1)+1)+1}=$$
$$\frac{1}{n(n+1)(n(n+1)+1)(n(n+1)(n(n+1)+1)+1)} \Rightarrow ▯$$

Xulosa

Ba’zi onlarning bo‘linish qoidalari, eng katta va eng kichik umumiy bo‘luvchilari yordamida qiziqarli mulohaza, tenglik va tengsizliklarni yechimlari ko‘rsatilgan.

Foydalanilgan adabiyotlar

1. A.V.Pogorelov, Analitik geometriya., T.O'qituvchi,, 1983

2. Искандаров. Олий алгебра.

3. Ж.Хожиев,А.С.Файнлейб, Алгебра ва сонлар назарияси, Тошкент-“Узбекистон”-2001.

4. Rajabov F.,Nurmatov A.,Analitik, geometriya va chizikli algebra, T.O'qituvchi, 1990y.

5. Некоторые решения, связанные с пределами и интегралами, А Ибрагимов, О Пулатов, С Тухтамуродов - Science and Education, 2024

6. Ayniyat yordamida tengsizliklarni isbotlash, A Ibragimov, O Pulatov, B Sag‘dullayeva, … - Science and Education, 2023

7. Tengsizliklarni yechishning noodatiy usuli, A Ibragimov, O Pulatov, I Teshaboyeva, A - Science and Education, 2023

8. Matematikadan olimpiada masalalari. H.Norjigitov, A.X.Nuraliyev. - Toshkent "Innovatsion rivojlanish nashriyoti - matbaa uyi" 2020. 244 bet