Bir va ikki o'lchamli ikki zarrachali diskret Shryodinger operatori xos qiymatlarining asimptotikasi

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Annotatsiya: Maqolada bir va ikki o'lchamli butun panjaralarda ikki zarrachali diskret Shryodinger operatorining muhim spektrning yuqori chegarasi yaqinidagi xos qiymati tahlil qilinadi. Tadqiqot nol masofali, rangi birga teng ixcham qo'zg'alishli model uchun olib boriladi. To'liq kvaziimpuls bo'yicha tolali ajralish qo'llanib, xos qiymat masalasi Birman–Schwinger prinsipi va Fredholm determinanti tenglamasiga keltiriladi. Kichik parametr sifatida bog'lanish doimiysining nolga intilishi tanlanadi. Asosiy natija shundan iboratki, bir o'lchamli holatda spektral chegaradan ajralish darajaviy tartibga, ikki o'lchamli holatda esa logarifmik xosmaslik tufayli eksponensial kichik tartibga ega bo'ladi. Natija spektral chegara yaqinida panjara o'lchamining xos qiymat dinamikasini keskin o'zgartirishini ko'rsatadi.

Kalit so'zlar: diskret Shryodinger operatori, ikki zarrachali operator, xos qiymat, muhim spektr, Birman-Schwinger prinsipi, Fredholm determinanti, assimptotika, panjara

Asymptotics of the eigenvalues of the one- and two-dimensional two-particle discrete Schrödinger operator

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Abstract: This paper studies a two-particle discrete Schrödinger operator on one- and two-dimensional integer lattices near the upper edge of the essential spectrum. The model contains a zero-range interaction represented by a compact rank-one perturbation. By means of the fiber decomposition with respect to the total quasi-momentum, the eigenvalue problem is reduced to a Birman-Schwinger equation and to the zero set of the corresponding Fredholm determinant. The coupling constant tending to zero is chosen as the small parameter. The main result states that the separation of the eigenvalue from the spectral edge is of power order in the one-dimensional case, while in the two-dimensional case the logarithmic threshold singularity yields an exponentially small asymptotic order. Hence the lattice dimension changes the threshold singularity and the eigenvalue dynamics.

Keywords: discrete Schrödinger operator, two-particle operator, eigenvalue, essential spectrum, Birman-Schwinger principle, Fredholm determinant, asymptotics, lattice

Kirish

Diskret Shryodinger operatorlari panjaradagi kvant tizimlarining spektral modelini beradi. Ikki zarrachali siljishga nisbatan invariant sistemada to'liq kvaziimpuls saqlanadi; shu sababli operator torus bo'yicha tolali operatorlar oilasiga ajraladi. Bunday ajratish ikki zarrachali panjara Gamiltonianlari spektrini, spektral chegara holatlarni va xos qiymatlarning joylashuvini o'rganishda asosiy apparatdir [2], [4], [7].

Mazkur ishda nol masofali o'zaro ta'sir tanlanadi. Impuls tasvirida bunday potensial har bir tolada rangi birga teng ixcham operator hosil qiladi. Natijada xos qiymat masalasi bitta skalyar Fredholm determinanti tenglamasiga keladi. Ushbu bir rangli mexanizm Lakaev-Holmatovning xos qiymatlar assimptotikasi haqidagi natijalari va Albeverio-Lakaev-Makarov-Muminovning spektral chegara effektlar tahlili bilan bevosita bog'liq [1], [4].

Spektral chegara yaqinidagi asosiy farq rezolventa integralining xosmasligidan kelib chiqadi:

$$\int \frac{dq}{z - E(q)}.$$

Bir o'lchamda bu integral teskari kvadrat ildiz tartibida, ikki o'lchamda esa logarifmik tartibda o'sadi. Shuning uchun ayni bir nol masofali qo'zg'alish bir o'lchamda darajaviy, ikki o'lchamda esa eksponensial kichik spektral ajralishga olib keladi [1], [3], [5].

Maqolaning maqsadi bir va ikki o'lchamli ikki zarrachali diskret Shryodinger operatori uchun muhim spektrning yuqori chegarasi yaqinidagi xos qiymatlarning assimptotik tartibini aniqlashdir. Kichik parametr bog'lanish doimiysining nolga intilishi sifatida tanlanadi:

$$\mu \downarrow 0.$$

Vazifalar quyidagilardan iborat: operator modelini aniqlash; muhim spektr chegarasini tavsiflash; Birman-Schwinger tenglamasini chiqarish; bir va ikki o'lchamdagi assimptotik natijalarni taqqoslash.

Metodlar

Ta'rif 1. Tolali ikki zarrachali diskret Shryodinger operatori. O'lcham quyidagi qiymatlardan birini oladi:

$$d \in \{1,2\}.$$

Furye almashtirishdan keyingi impuls fazosi va normallashtirilgan o‘lchov quyidagicha olinadi:

$$\mathbb{T}^d = (-\pi, \pi]^d, \qquad L^2(\mathbb{T}^d), \qquad d\eta(q) = (2\pi)^{-d}dq.$$

Bir zarrachali dispersiya va to‘liq kvaziimpulsga mos erkin energiya funksiyasi:

$$\varepsilon(p) = \sum_{j=1}^{d}\left(1 - \cos p_j\right), \qquad E_K(q) = \varepsilon(q) + \varepsilon(K - q).$$

Erkin tolali operator va nol masofali potensial impuls tasvirida quyidagicha aniqlanadi:

$$(h_0(K)f)(q) = E_K(q)f(q), \qquad (Vf)(q) = \int_{\mathbb{T}^d} f(t)d\eta(t).$$

To‘liq tolali operator:

$$h_\mu(K) = h_0(K) + \mu V, \qquad \mu > 0.$$

Potensial operator rangi birga teng, ixcham va o‘z-o‘ziga qo‘shma. Shuning uchun to‘liq operator ham o‘z-o‘ziga qo‘shma bo‘ladi. Bunday model nol masofali ikki zarrachali diskret Shryodinger operatorlarining maxsus, ammo spektral chegara tahlili uchun asosiy holatidir [1], [2].

Ta’rif 2. Muhim spektr chegarasi. Erkin energiya funksiyasining ekstremal qiymatlari:

$$m(K) = \min_{q\in\mathbb{T}^d} E_K(q), \qquad M(K) = \max_{q\in\mathbb{T}^d} E_K(q).$$

Lemma 1. Har bir sobit to‘liq kvaziimpuls uchun quyidagi tengliklar bajariladi:

$$\sigma\big(h_0(K)\big) = [m(K), M(K)], \qquad \sigma_{\text{ess}}\left(h_\mu(K)\right) = [m(K), M(K)].$$

Isbot. Erkin operator ko‘paytirish operatoridir; uning spektri uzluksiz energiya funksiyasining ixcham torusdagi qiymatlar oralig‘iga teng. Potensial operator chekli rangli bo‘lgani sababli ixcham. Weyl ixcham qo‘zg‘alish teoremasi muhim spektr invariantligini beradi [8].

Ta’rif 3. Birman-Schwinger operatori. Yuqori spektral chegara ustidagi parametr qaraladi:

$$z > M(K).$$

Rangi birga teng holatda Birman-Schwinger operatorining yagona nol bo‘lmagan xos qiymati va Fredholm determinanti quyidagicha yoziladi:

$$\beta_\mu(K, z) = \mu I_d(K, z), \qquad I_d(K, z) = \int_{\mathbb{T}^d} \frac{d\eta(q)}{z - E_K(q)},$$

$$\Delta_\mu(K, z) = 1 - \mu I_d(K, z).$$

Lemma 2. Spektral parametr yuqori chegara ustida bo‘lsa, u xos qiymat bo‘lishi uchun quyidagi tenglama zarur va yetarlidir:

$$\Delta_\mu(K, z) = 0.$$

Isbot. Xos qiymat tenglamasi quyidagicha yoziladi:

$$(E_K(q)-z)f(q)+\mu\int_{\mathbb{T}^d} f(t)d\eta(t)=0.$$

Agar nol bo'lmagan yechim mavjud bo'lsa, rezolventa orqali

$$f(q)=\frac{\mu C}{z-E_K(q)},\qquad C=\int_{\mathbb{T}^d} f(t)d\eta(t)$$

munosabat olinadi. Uni integrallash skalyar Birman-Schwinger tenglamasini beradi:

$$1=\mu\int_{\mathbb{T}^d}\frac{d\eta(q)}{z-E_K(q)}.$$

Bu aynan Fredholm determinantining noliga teng [1], [4].

Asimptotik tahlilda nol kvaziimpuls olinadi:

$$K=0,\qquad M(0)=4d,\qquad q_0=(\pi,\dots,\pi).$$

Spektral masofa quyidagicha belgilanadi:

$$\delta=z-M(0)>0.$$

Maksimum nuqta atrofida energiya funksiyasi kvadratik ko'rinishga ega:

$$M(0)-E_0(q)=|q-q_0|^2+O(|q-q_0|^4).$$

Shuning uchun asosiy chegaraviy integrallar quyidagicha bo'ladi:

$$I_1(0,M(0)+\delta)=\frac{1}{2\sqrt{\delta}}+O(1),\qquad \delta\downarrow 0,$$

$$I_2(0,M(0)+\delta)=\frac{1}{4\pi}\ln\frac{1}{\delta}+b_0+o(1),\qquad \delta\downarrow 0.$$

Natijalar va muhokama

Asosiy farazlar: qo'zg'alish nol masofali, rangi birga teng, musbat ishorali va ixcham; to'liq kvaziimpuls nolga teng; xos qiymat muhim spektrning yuqori chegarasi ustida qaraladi. Spektral ajralish quyidagicha belgilanadi:

$$\delta_\mu=z_\mu-M(0).$$

Fredholm determinant tenglamasi:

$$1=\mu I_d\big(0,M(0)+\delta_\mu\big).$$

Integral monoton kamayadi, chunki

$$\frac{d}{dz}I_d(0,z)=-\int_{\mathbb{T}^d}\frac{d\eta(q)}{\big(z-E_0(q)\big)^2}<0.$$

Shuning uchun yuqori chegara ustida ildiz yagona. Bu holat ikki zarrachali diskret operatorlarning xos qiymatlar soni va joylashuvi haqidagi natijalar bilan mos keladi [2], [6].

Teorema 1. Bir o'lchamli panjara uchun nol masofali musbat qo'zg'alish berilgan bo'lsin. U holda bog'lanish parametri nolga intilganda muhim spektrning yuqori chegarasi ustidagi yagona xos qiymat quyidagi assimptotik munosabatni qanoatlantiradi:

$$z_\mu - M(0) = \frac{\mu^2}{4} + o(\mu^2), \qquad \mu \downarrow 0.$$

Isbot. Bir o'lchamda maksimum nuqta atrofidagi kvadratik yoyilma rezolventa integraliga quyidagi asosiy hadni beradi:

$$I_1(0, M(0) + \delta) = \frac{1}{2\sqrt{\delta}} + O(1).$$

Birman-Schwinger tenglamasiga qo'yib,

$$1 = \mu\left(\frac{1}{2\sqrt{\delta_\mu}} + O(1)\right)$$

olinadi. Bundan

$$\sqrt{\delta_\mu} = \frac{\mu}{2} + o(\mu)$$

va teorema formulasi kelib chiqadi. Darajaviy tartib Lakaev-Holmatovning nol masofali model uchun spektral chegara assimptotikasi bilan bir xil integral xosmaslikka tayanadi [1].

Teorema 2. Ikki o'lchamli panjara uchun nol masofali musbat qo'zg'alish berilgan bo'lsin. U holda bog'lanish parametri nolga intilganda muhim spektrning yuqori chegarasi ustidagi yagona xos qiymat quyidagi ko'rinishga ega:

$$z_\mu - M(0) = C_* \exp\left(-\frac{4\pi}{\mu}\right)\big(1 + o(1)\big), \qquad \mu \downarrow 0,$$

bu yerda

$$C_* = \exp(4\pi b_0) > 0.$$

Isbot. Ikki o'lchamda mahalliy integralning xosmas qismi logarifmik:

$$I_2(0, M(0) + \delta) = \frac{1}{4\pi}\ln\frac{1}{\delta} + b_0 + o(1).$$

Determinant tenglamasi shuning uchun

$$1 = \mu\left(\frac{1}{4\pi}\ln\frac{1}{\delta_\mu} + b_0 + o(1)\right)$$

ko'rinishga keladi. Bundan

$$\ln\frac{1}{\delta_\mu} = \frac{4\pi}{\mu} - 4\pi b_0 + o(1)$$

va ko'rsatkichli shaklga o'tkazish orqali teorema formulasi olinadi. Ikki o'lchamli spektral chegara integralining logarifmik tabiati delta potensialli diskret Shryodinger operatorlari tahlilida ham asosiy rol o'ynaydi [5].

Taqqoslash quyidagi ikki determinant tenglamasida jamlanadi:

$$1 \sim \frac{\mu}{2\sqrt{\delta_\mu}} \quad \text{bir o'lchamda,} \qquad 1 \sim \frac{\mu}{4\pi}\ln\frac{1}{\delta_\mu} \quad \text{ikki o'lchamda.}$$

Shu sababli spektral ajralishlar turlicha tartibga ega:

$$\delta_\mu \asymp \mu^2 \quad \text{bir o‘lchamda,} \qquad \delta_\mu \asymp \exp(-4\pi/\mu) \quad \text{ikki o‘lchamda.}$$

O‘lcham oshishi bilan spektral chegara integralining kuchli algebraik xosmasligi logarifmik xosmaslikka almashadi. Natijada Birman-Schwinger tenglamasining teskari yechimi algebraik formuladan eksponensial formulaga o‘tadi. Keltirilgan teoremalar umumiy qisqa ta’sirli potensiallar uchun emas, balki rangi birga teng nol masofali qo‘zg‘alish uchun yozildi; shunga qaramay asosiy o‘lchamga bog‘liq tartib erkin energiyaning ekstremum geometriyasi bilan belgilanadi [1], [3], [4].

Xulosa

Maqolada bir va ikki o‘lchamli ikki zarrachali diskret Shryodinger operatori nol masofali musbat ixcham qo‘zg‘alish holida qaraldi. To‘liq kvaziimpuls bo‘yicha tolali ajralish yordamida xos qiymat masalasi Birman-Schwinger prinsipi orqali skalyar Fredholm determinant tenglamasiga keltirildi.

Bir o‘lchamda rezolventa integrali teskari kvadrat ildiz tartibida o‘sadi va xos qiymatning yuqori chegaradan ajralishi quyidagi ko‘rinishga ega:

$$z_\mu - M(0) = \frac{\mu^2}{4} + o(\mu^2).$$

Ikki o‘lchamda logarifmik xosmaslik yuzaga keladi va xos qiymatning ajralishi eksponensial kichik tartibda yoziladi:

$$z_\mu - M(0) = C_* \exp\left(-\frac{4\pi}{\mu}\right)\big(1 + o(1)\big).$$

Demak, panjara o‘lchami Fredholm determinantining spektral chegara yoyilmasini o‘zgartiradi va bu o‘zgarish xos qiymatlarning muhim spektr chegarasiga yaqinlashish tezligida aks etadi.

Foydalanilgan adabiyotlar

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