On certain classes of A(z)-analytic functions defined by the noor integral operator

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Abstract: We study $A(z)$-analytic functions, i.e. solutions of the Beltrami equation $f_{\bar{z}} = A(z)f_z$, where $A(z)$ is antianalytic in a convex domain D and $|A(z)| \leq c < 1$. The unit disc is replaced by the $A(z)$-lemniscate $\mathcal{L} = \{z \in D: |\psi(z)| < 1\}$, $\psi(z) = z + \int_0^z \overline{A(\tau)}\, d\tau$. We introduce $A(z)$-analogues of the Hadamard product, of the Ruscheweyh derivative and of the Noor integral operator, and define the classes $R^{\star}_{k,A}(n,\alpha)$ related to functions of bounded boundary rotation. First we prove a transfer principle: ψ maps $\mathcal{L}$ diffeomorphically onto the unit disc, and f is $A(z)$-analytic in $\mathcal{L}$ if and only if $f \circ \psi^{-1}$ is holomorphic. On this basis we obtain an inclusion theorem with the sharp order α_n, invariance of the classes under $A(z)$-convolution with $A(z)$-convex functions and under $A(z)$-analogues of the Alexander, Libera and Bernardi operators, coefficient and growth estimates, a radius problem, and a Carleman-type formula which recovers f from the values of $I_n f$ on a part of the boundary. The proofs rely on $A(z)$-analogues of Jack's lemma, of the Ruscheweyh-Sheil-Small theorem and of the Carleman formula.

Keywords: $A(z)$-analytic function, $A(z)$-lemniscate, Noor integral operator, $A(z)$-convolution, bounded boundary rotation, starlike and convex functions, Carleman formula

1. Introduction

Let $E = \{w \in \mathbb{C}: |w| < 1\}$ and let $\mathcal{A}$ denote the family of holomorphic functions $F(w) = w + \sum_{m\geq 2} a_m w^m$ in E. For $0 \leq \alpha < 1$ the class $P(\alpha)$ consists of holomorphic p in E with $p(0) = 1$ and $\mathrm{Re}\, p > \alpha$. For $k \geq 2$ the class $P_k(\alpha)$, introduced for $\alpha = 0$ by Pinchuk [9], is formed by the functions

$$h = \left(\frac{k}{4} + \frac{1}{2}\right) p_1 - \left(\frac{k}{4} - \frac{1}{2}\right) p_2, \qquad p_1, p_2 \in P(\alpha). \qquad (1)$$

A function $F \in \mathcal{A}$ is in $R^{\star}_k(\alpha)$ when $wF'/F \in P_k(\alpha)$, and in $V_k(\alpha)$ when $(wF')'/F' \in P_k(\alpha)$. Thus $V_k(0)$ is the class of functions of bounded boundary rotation, while $R^{\star}_2(\alpha) = S^{\star}(\alpha)$ and $V_2(\alpha) = C(\alpha)$ are the classical starlike and convex functions of order α [10].

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With $f_n(w) = w(1-w)^{-n-1}$, $n \in \mathbb{N}_0$, and the Hadamard product $\star$, the Ruscheweyh derivative is $D^n F = f_n \star F$ [11]. Choosing $f_n^{(-1)}$ so that $f_n \star f_n^{(-1)} = w(1-w)^{-2}$, one obtains the Noor integral operator [6, 7]

$$I_n F = f_n^{(-1)} \star F = w + \sum_{m \geq 2} \frac{n!\ m!}{(m+n-1)!}\, a_m w^m, \quad (2)$$

for which $I_0 F = wF'$ and $I_1 F = F$. K. I. Noor and M. A. Noor [8] considered the class $R_k^\star(n, \alpha)$ of those $F \in \mathcal{A}$ for which $I_n F \in R_k^\star(\alpha)$, and obtained inclusion relations, convolution properties, coefficient bounds and a radius result; related questions were treated in [3,4].

The aim of this paper is to develop the same circle of questions for $A(z)$-analytic functions, that is, for solutions of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = A(z)\, \frac{\partial f}{\partial z} \quad (3)$$

in the case when $A(z)$ is antianalytic and $|A(z)| \leq c < 1$. For this case Sadullaev and Jabborov [13] proved the Cauchy theorem and the Cauchy integral formula, constructed the Taylor and Laurent expansions in $A(z)$-lemniscates and obtained analogues of the Picard and Montel theorems. The $A(z)$-harmonic functions, the Poisson formula in lemniscates, the Hardy class H_A^1 and the Fatou theorem were used by Zhabborov, Narzillaev and Husenov [17] and by Zhabborov and Husenov [15,16] to derive Carleman-type formulas for $A(z)$-analytic functions. In accordance with these papers the role of the unit disc is played here by the $A(z)$-lemniscate $L(0,1)$.

Several classical tools that are used below are discussed in [18-23]. Euler's beta and gamma integrals and their applications [18] lead to the integral representation of the $A(z)$-Noor operator (Proposition 4.5(b)). The Poisson kernel and the Herglotz-type representation of Section 4 are expansions in trigonometric Fourier series [22], and in the plane they are the simplest case of the theory of spherical functions [21]. The notion of bounded boundary rotation rests on functions of bounded total variation [23]. Relations of the form $\psi\ \partial_z h + nh = (n+1)g$, which appear in Lemma 5.4 and Theorem 6.1, are first-order differential equations, and the classification of equilibrium points of autonomous differential equations [19] gives useful background for their qualitative study. Finally, the system $\{\psi^m\}_{m \geq 0}$ is orthonormal on $\partial\mathcal{L}$ with respect to the measure $|d\psi|/(2\pi)$, so its Gram determinants [20] are equal to 1; this explains why the coefficients in Theorem 6.5 are obtained by the same formulas as in the disc.

Our approach differs from the classical one in [8]. Instead of decompositions of functions of bounded boundary rotation, we first prove that the function ψ is a bi-Lipschitz diffeomorphism which maps the lemniscate onto the disc and converts the operator $\partial/\partial z$ into the ordinary derivative (Theorem 3.3). The basic tools are then the

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$A(z)$-version of Jack's lemma, the $A(z)$-version of the Ruscheweyh-Sheil-Small convolution theorem, and the $A(z)$-Carleman formula.

2. Preliminaries

Throughout the paper $D \subset \mathbb{C}$ is a convex domain containing 0, and $A(z)$ is antianalytic in D (i.e. $\partial A/\partial z = 0$) with

$$|A(z)| \le c < 1, \qquad z \in D. \qquad (4)$$

A function $f \in C^1(G)$, $G \subset D$, satisfying (3) in G is called $A(z)$-analytic in G; the set of such functions is denoted by $\mathcal{O}_A(G)$. Because A is real-analytic, $\mathcal{O}_A(G) \subset C^\infty(G)$ [14, 13]. Equation (3) is linear and the operator $\partial_{\bar z} - A\,\partial_z$ obeys the Leibniz rule, hence $\mathcal{O}_A(G)$ is closed under sums and products, and $1/f \in \mathcal{O}_A(G)$ wherever $f \neq 0$.

Theorem 2.1 ([13]). *If $f \in \mathcal{O}_A(G) \cap C(\overline{G})$ and ∂G is rectifiable, then $\int_{\partial G} f\,(z)\,(dz + A(z)\,d\bar z) = 0$.*

Since $\overline{A}$ is holomorphic in the convex domain D, the integral of $\overline{A(\tau)}\,d\tau$ along a path $\gamma(a,z) \subset D$ depends only on the end points. Set

$$\psi(z,a) = z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)}\,d\tau}, \qquad \psi(z) := \psi(z,0). \qquad (5)$$

Then $\partial\psi/\partial z = 1$, $\partial\psi/\partial\bar z = A(z)$, so $\psi(\cdot,a) \in \mathcal{O}_A(D)$, and

$$d\psi(z,a) = dz + A(z)\,d\bar z, \qquad \psi(\zeta,z) = \psi(\zeta,a) - \psi(z,a). \qquad (6)$$

The function $K(\zeta,z) = \frac{1}{2\pi i\,\psi(\zeta,z)}$ is the $A(z)$-Cauchy kernel [13], and

$$L(a,r) = \{z \in D\colon\ |\psi(z,a)| < r\}$$

is the $A(z)$-lemniscate with centre a and radius r.

Theorem 2.2 ([13]). *Let $G \subset\subset D$ have piecewise smooth boundary and $f \in \mathcal{O}_A(G) \cap C(\overline{G})$. Then $f(z) = \int_{\partial G} K(\zeta,z) f(\zeta)\big(d\zeta + A(\zeta)d\bar\zeta\big)$, $z \in G$. If $f \in \mathcal{O}_A\big(L(a,r)\big)$ and $L(a,r) \subset\subset D$, then*

$$f(z) = \sum_{k=0}^{\infty} c_k\,\psi^k(z,a), c_k = \frac{1}{2\pi i}\int_{\partial L(a,\rho)} \frac{f(\zeta)\,\big(d\zeta + A(\zeta)\,d\bar\zeta\big)}{\psi(\zeta,a)^{k+1}}, \qquad (7)$$

where $0 < \rho < r$.

The real part of an $A(z)$-analytic function is $A(z)$-harmonic and is reproduced in a lemniscate by the Poisson integral [15,17]

$$u(z) = \frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r} u\,(\zeta)\,\frac{r^2 - |\psi(z,a)|^2}{|\psi(\zeta,z)|^2}\,\big|d\zeta + A(\zeta)\,d\bar\zeta\big|. \qquad (8)$$

The Hardy class $H^1_A\big(L(a,r)\big)$ consists of $f \in \mathcal{O}_A\big(L(a,r)\big)$ whose integral means $\frac{1}{2\pi\rho}\int_{|\psi(z,a)|=\rho} |f|\,|dz + A\,d\bar z|$ are bounded for $\rho < r$; such functions have angular boundary values f^* almost everywhere [16,17].

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Theorem 2.3 ([15,17]). *Let $L(a,r) \subset\subset D$, let $M \subset \partial L(a,r)$ be of positive measure, ω the $A(z)$-harmonic measure of M, v its $A(z)$-conjugate and $\varphi = e^{-(\omega+iv)}$. Then for $f \in H_A^1\big(L(a,r)\big)$*

$$f(z) = \frac{1}{2\pi i} \lim_{m\to\infty} \int_M f^*(\zeta) \left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m \frac{d\zeta + A(\zeta)\, d\bar{\zeta}}{\psi(\zeta, z)},$$

uniformly on compact subsets of $L(a,r)$.

3. The lemniscate as a model domain

Lemma 3.1. *For all $z_1, z_2 \in D$*

$$(1-c)\,|z_1 - z_2| \le |\psi(z_1) - \psi(z_2)| \le (1+c)\,|z_1 - z_2|. \qquad (9)$$

Hence ψ is a C^∞ diffeomorphism of D onto the open set $\psi(D)$ with Jacobian $1 - |A|^2 \ge 1 - c^2$, and $(1-c)|z| \le |\psi(z)| \le (1+c)|z|$.

Proof. Integrating along the segment $[z_2, z_1] \subset D$ we get $\psi(z_1) - \psi(z_2) = (z_1 - z_2) + \overline{J}$, where $J = \int_{[z_2,z_1]} \overline{A(\tau)}\, d\tau$ and, by (4), $|J| \le c|z_1 - z_2|$. The triangle inequality gives (9). The Jacobian of ψ is $|\psi_z|^2 - |\psi_{\bar{z}}|^2 = 1 - |A|^2 > 0$; an injective map with non-vanishing Jacobian is a diffeomorphism onto its image, which is open. Putting $z_2 = 0$ completes the proof. ▫

In what follows we fix

$$\mathcal{L} := L(0,1) = \{z \in D\colon |\psi(z)| < 1\}$$

and assume $\mathcal{L} \subset\subset D$. By Lemma 3.1, $\{|z| < (1+c)^{-1}\} \subset \mathcal{L} \subset \{|z| < (1-c)^{-1}\}$, so this assumption holds whenever D contains the closed disc $|z| \le (1-c)^{-1}$.

Lemma 3.2. *The map ψ is a diffeomorphism of $\mathcal{L}$ onto E, of $\overline{\mathcal{L}}$ onto $\overline{E}$, and $\psi\big(L(0,\rho)\big) = \{|w| < \rho\}$ for $0 < \rho \le 1$.*

Proof. The set $\psi(\mathcal{L}) = \psi(D) \cap E$ is open in E. Let $w_j = \psi\big(z_j\big) \to w \in E$ with $z_j \in \mathcal{L}$. Since $\overline{\mathcal{L}}$ is a compact subset of D, a subsequence of z_j tends to some $z \in D$, and $\psi(z) = w$ with $|w| < 1$, i.e. $z \in \mathcal{L}$. Thus $\psi(\mathcal{L})$ is also closed in the connected set E and contains 0, so $\psi(\mathcal{L}) = E$. The other assertions are proved in the same way. ▫

In the coordinate $w = \psi(z)$ the curves $\gamma_\zeta = \psi^{-1}([0, \psi(\zeta)])$ play the role of radii, and the level lines $\partial L(0,\rho)$ the role of circles (cf. [17]). For $|x| \le 1$ we shall use the $A(z)$-dilation

$$\omega_x(z) = \psi^{-1}\big(x\,\psi(z)\big), \qquad z \in \mathcal{L}, \qquad (10)$$

which maps $\mathcal{L}$ into itself, and for $|x| = 1$ is an automorphism of $\mathcal{L}$.

Theorem 3.3 (Transfer principle). *Let $f \in C^1(\mathcal{L})$ and $F = f \circ \psi^{-1}$ in E. Then*

$$\frac{\partial f}{\partial \bar{z}} - A\,\frac{\partial f}{\partial z} = (1 - |A|^2)\left(\frac{\partial F}{\partial \bar{w}} \circ \psi\right). \qquad (11)$$

Consequently $f \in \mathcal{O}_A(\mathcal{L})$ if and only if F is holomorphic in E, and then

$$\partial_z^j f = F^{(j)} \circ \psi, \qquad j = 1,2,\ldots \qquad (12)$$

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In particular $\partial_z f \in \mathcal{O}_A(\mathcal{L})$*, and the coefficients* c_k *in (7) with* $a = 0$ *are the Taylor coefficients of* F*.*

Proof. We have $\psi_z = 1, \psi_{\bar{z}} = A, \overline{\psi}_z = \overline{A}, \overline{\psi}_{\bar{z}} = 1$. The chain rule applied to $f = F \circ \psi$ gives $f_z = F_w + \overline{A}\, F_{\overline{w}}$ and $f_{\bar{z}} = A\, F_w + F_{\overline{w}}$, which yields (11). Since $1 - |A|^2 > 0$, f satisfies (3) exactly when $F_{\overline{w}} = 0$. If F is holomorphic, then $f_z = F' \circ \psi$; as $F' \circ \psi$ has the same structure, induction gives (12). Finally, the change of variable $w = \psi(\zeta)$, $dw = d\zeta + A\, d\bar{\zeta}$, maps $\partial L(0, \rho)$ onto $|w| = \rho$ and turns c_k into the Cauchy coefficient of F. ▫

Remark 3.4. By (12), ∂_z is the derivative with respect to the variable ψ; for instance $\partial_z \psi^m = m\psi^{m-1}$. All results below remain true in any lemniscate $L(a, r) \subset \subset D$ if $\psi(z)$ is replaced by $\psi(z, a)/r$.

Every $f \in \mathcal{O}_A(\mathcal{L})$ with $f(0) = 0$ and $\partial_z f(0) = 1$ can be written as

$$f(z) = \psi(z) + \sum_{m=2}^{\infty} a_m\, \psi^m(z), \qquad z \in \mathcal{L}. \qquad (13)$$

We denote this family by $\mathcal{A}_A$; thus $f \in \mathcal{A}_A$ if and only if $f \circ \psi^{-1} \in \mathcal{A}$.

4. The classes $\boldsymbol{R^{\star}_{k,A}(n, \alpha)}$

Definition 4.1. Let $0 \le \alpha < 1$ and $k \ge 2$.

(i) $P_A(\alpha)$ is the class of $p \in \mathcal{O}_A(\mathcal{L})$ with $p(0) = 1$ and $\mathrm{Re}\, p > \alpha$ in $\mathcal{L}$; $P_{k,A}(\alpha)$ is the class of functions $h = \left(\frac{k}{4} + \frac{1}{2}\right) p_1 - \left(\frac{k}{4} - \frac{1}{2}\right) p_2$ with $p_1, p_2 \in P_A(\alpha)$.

(ii) $f \in \mathcal{A}_A$ is *$A(z)$-starlike of order* α, $f \in S^{\star}_A(\alpha)$, if $\mathrm{Re}\frac{\psi\, \partial_z f}{f} > \alpha$ in $\mathcal{L}$, and *$A(z)$-convex of order* α, $f \in C_A(\alpha)$, if $\mathrm{Re}\left(1 + \frac{\psi\, \partial_z^2 f}{\partial_z f}\right) > \alpha$ in $\mathcal{L}$.

(iii) $f \in R^{\star}_{k,A}(\alpha)$ if $\frac{\psi\, \partial_z f}{f} \in P_{k,A}(\alpha)$, and $f \in V_{k,A}(\alpha)$ if $1 + \frac{\psi\, \partial_z^2 f}{\partial_z f} \in P_{k,A}(\alpha)$.

Since by (12) $\psi\, \partial_z f = (wF') \circ \psi$ and $\psi\, \partial_z^2 f = (wF'') \circ \psi$, Theorem 3.3 gives

$$p \in P_{k,A}(\alpha) \Leftrightarrow p \circ \psi^{-1} \in P_k(\alpha), f \in R^{\star}_{k,A}(\alpha) \Leftrightarrow F \in R^{\star}_k(\alpha), f \in V_{k,A}(\alpha) \Leftrightarrow F \in V_k(\alpha). (14)$$

Moreover $f \in V_{k,A}(\alpha) \Leftrightarrow \psi\, \partial_z f \in R^{\star}_{k,A}(\alpha)$, $R^{\star}_{2,A}(\alpha) = S^{\star}_A(\alpha)$ and $V_{2,A}(\alpha) = C_A(\alpha)$.

Proposition 4.2 (Herglotz-type representation). *A function* p *belongs to* P_A *if and only if there is a probability measure* μ *on* $[0, 2\pi)$ *with*

$$p(z) = \int_0^{2\pi} \frac{e^{it} + \psi(z)}{e^{it} - \psi(z)}\, d\mu(t), \qquad z \in \mathcal{L}. \qquad (15)$$

Proof. Suppose first that p is continuous on $\overline{\mathcal{L}}$. Then $u = \mathrm{Re}\, p$ satisfies the Poisson formula (8) with $a = 0$, $r = 1$. On $\partial\mathcal{L}$ put $\psi(\zeta) = e^{it}$; by (6), $\left|d\zeta + A\, d\bar{\zeta}\right| = |d\psi(\zeta)| = dt$ and $\psi(\zeta, z) = e^{it} - \psi(z)$, so the Poisson kernel is $\mathrm{Re}\frac{e^{it} + \psi(z)}{e^{it} - \psi(z)}$. The right

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side of (15) with $d\mu = \frac{1}{2\pi} u\, dt$ is $A(z)$-analytic, has real part u and equals 1 at $z = 0$; two $A(z)$-analytic functions with equal real parts and equal values at a point coincide (by (11) their difference is a holomorphic function of w with zero real part), so (15) holds. In general we apply this to $p \circ \omega_\rho$, $\rho < 1$, and pass to the limit $\rho \to 1$ with the help of Helly's selection theorem. The converse is clear, since the integrand has positive real part. ▫

Definition 4.3. For $f = \sum_{m\geq 1} a_m \psi^m$ and $g = \sum_{m\geq 1} b_m \psi^m$ in $\mathcal{O}_A(\mathcal{L})$ the $A(z)$-*convolution* is

$$(f \star_A g)(z) = \sum_{m=1}^{\infty} a_m\, b_m\, \psi^m(z), \qquad z \in \mathcal{L}.$$

By Theorem 3.3, $f \star_A g = (F \star G) \circ \psi$; hence $\star_A$ is commutative and associative, $\psi/(1-\psi)$ is its unit on $\mathcal{A}_A$, and $f \star_A g \in \mathcal{O}_A(\mathcal{L})$. Termwise integration of (7) gives, for $|\psi(z)| < \rho < 1$,

$$(f \star_A g)(z) = \frac{1}{2\pi i} \int_{\partial L(0,\rho)} f(\zeta)\; g\left(\psi^{-1}\left(\frac{\psi(z)}{\psi(\zeta)}\right)\right) \frac{d\zeta + A(\zeta)\, d\bar{\zeta}}{\psi(\zeta)}. \qquad (16)$$

Definition 4.4. Let $f_{n,A} = \psi(1-\psi)^{-n-1}$ and let $f_{n,A}^{(-1)}$ be defined by $f_{n,A} \star_A f_{n,A}^{(-1)} = \psi(1-\psi)^{-2}$, $n \in \mathbb{N}_0$. For $f \in \mathcal{A}_A$ of the form (13) put

$$D_A^n f = f_{n,A} \star_A f = \psi + \sum_{m\geq 2} \frac{(m+n-1)!}{n!\,(m-1)!}\, a_m \psi^m, \qquad I_A^n f = f_{n,A}^{(-1)} \star_A f$$

$$= \psi + \sum_{m\geq 2} c_m(n)\, a_m \psi^m, \qquad (17)$$

where $c_m(n) = \frac{n!\,m!}{(m+n-1)!}$. We call D_A^n the $A(z)$-Ruscheweyh derivative and I_A^n the $A(z)$-Noor integral operator.

By (12) and the classical formula $D^n F = w(w^{n-1}F)^{(n)}/n!$ we have $D_A^n f = \psi\; \partial_z^n(\psi^{n-1} f)/n!$. Also $I_A^0 f = \psi\; \partial_z f$, $I_A^1 f = f$ and $I_A^n f = (I_n F) \circ \psi$.

Proposition 4.5. *Let $f \in \mathcal{A}_A$ and $n \in \mathbb{N}_0$.*

(a) $(n+1)\, I_A^n f - n\, I_A^{n+1} f = \psi\; \partial_z(I_A^{n+1} f)$.

(b) For $n \geq 1$,

$$I_A^n f(z) = n \int_0^1 (1-t)^{n-1}\; \psi(z)\, (\partial_z f)\big(\omega_t(z)\big)\, dt, \qquad z \in \mathcal{L}. \qquad (18)$$

(c) $D_A^n(I_A^n f) = \psi\; \partial_z f$, *and for* $n \geq 1$

$$f = \frac{1}{n!}\, \partial_z^{n-1}(\psi^{n-1}\, I_A^n f). \qquad (19)$$

Proof. (a) From $c_m(n)/c_m(n+1) = (m+n)/(n+1)$ we get $(n+1)c_m(n) - n\, c_m(n+1) = m\, c_m(n+1)$, and $\psi\; \partial_z \psi^m = m\psi^m$.

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(b) We have $I_nF = [w\,{}_2F_1(2,1;n+1;w)] \star F$ and, by Euler's integral, ${}_2F_1(2,1;n+1;w) = n\int_0^1 (1-t)^{n-1}\,(1-tw)^{-2}dt$. Since $\frac{w}{(1-tw)^2} \star F = wF'(tw)$, we get $I_nF(w) = n\int_0^1 (1-t)^{n-1}\,wF'(tw)\,dt$. Now put $w = \psi(z)$ and note $F'\big(t\psi(z)\big) = (\partial_z f)\big(\omega_t(z)\big)$ by (10), (12).

(c) $f_n \star f_n^{(-1)} \star F = \frac{w}{(1-w)^2} \star F = wF'$. If $G = I_nF = \sum c_m(n)a_m w^m$, the coefficient of w^m in $\frac{1}{n!}(w^{n-1}G)^{(n-1)}$ equals $\frac{(m+n-1)!}{n!\,m!}\,c_m(n)\,a_m = a_m$. It remains to apply Theorem 3.3. ▫

Definition 4.6. Let $k \geq 2$, $0 \leq \alpha < 1$, $n \in \mathbb{N}_0$. A function $f \in \mathcal{A}_A$ belongs to $R^{\star}_{k,A}(n,\alpha)$ if $I_A^n f \in R^{\star}_{k,A}(\alpha)$, i.e.

$$\frac{\psi(z)\;\partial_z\big(I_A^n f(z)\big)}{I_A^n f(z)} \in P_{k,A}(\alpha), \qquad z \in \mathcal{L}.$$

We write $R^{\star}_{k,A}(n) = R^{\star}_{k,A}(n,0)$. Then $R^{\star}_{k,A}(0,\alpha) = V_{k,A}(\alpha)$, $R^{\star}_{k,A}(1,\alpha) = R^{\star}_{k,A}(\alpha)$, and by (14)

$$f \in R^{\star}_{k,A}(n,\alpha) \Leftrightarrow f \circ \psi^{-1} \in R^{\star}_{k}(n,\alpha). \qquad (20)$$

Example 4.7. Let $D = \{|z| < 2\}$ and $A(z) = \tilde{z}/8$. Then $c = 1/4$, $\psi(z) = z + \tilde{z}^{\,2}/16$, and since $(1-c)^{-1} = 4/3 < 2$, the lemniscate $\mathcal{L} = \{|z + \tilde{z}^2/16| < 1\}$ is compactly contained in D. The function $f \in \mathcal{A}_A$ determined by

$$I_A^n f = \frac{\psi\,(1-\delta_1\psi)^{(k/2-1)(1-\alpha)}}{(1-\delta_2\psi)^{(k/2+1)(1-\alpha)}}, \qquad |\delta_1| = |\delta_2| = 1,$$

belongs to $R^{\star}_{k,A}(n,\alpha)$ and is not holomorphic in z. For $k = 2$, $\alpha = 0$, $n = 1$ it is the $A(z)$-Koebe function $\psi/(1-\delta_2\psi)^2$.

5. Lemmas

Lemma 5.1 ($A(z)$-analogue of Jack's lemma). *Let* $\varpi \in \mathcal{O}_A(\mathcal{L})$, $\varpi(0) = 0$, $\varpi \not\equiv 0$. *If* $|\varpi|$ *restricted to the closed lemniscate* $\overline{L(0,r)}$, $0 < r < 1$, *attains its maximum at* $z_0 \in \partial L(0,r)$, *then* $\psi(z_0)\;\partial_z\varpi(z_0) = \kappa\,\varpi(z_0)$ *with a real* $\kappa \geq 1$.

Proof. Let $W = \varpi \circ \psi^{-1}$ and $w_0 = \psi(z_0)$. By Lemma 3.2 and Theorem 3.3, W is holomorphic in E, $W(0) = 0$, and $|W|$ attains its maximum over $|w| \leq r$ at w_0, $|w_0| = r$. By Jack's lemma [2], $w_0W'(w_0) = \kappa W(w_0)$ with $\kappa \geq 1$, and by (12) $w_0W'(w_0) = \psi(z_0)\,\partial_z\varpi(z_0)$. ▫

Lemma 5.2. *Let* $n \in \mathbb{N}_0$, $p \in \mathcal{O}_A(\mathcal{L})$, $p(0) = 1$, *and*

$$\mathrm{Re}\left[p(z) + \frac{\psi(z)\;\partial_z p(z)}{p(z)+n}\right] > 0, \qquad z \in \mathcal{L}. \qquad (21)$$

Then $\mathrm{Re}\,p(z) > \alpha_n$ *in* $\mathcal{L}$, *where*

$$\alpha_n = \frac{2}{(2n+1) + \sqrt{4n^2+4n+9}} \in \left(0, \frac{1}{2}\right]. \qquad (22)$$

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Proof. Put $\alpha = \alpha_n$ and define ϖ by

$$p = \frac{1 + (1-2\alpha)\varpi}{1-\varpi}, \qquad \text{i.e.} \qquad \varpi = \frac{p-1}{p+1-2\alpha}.$$

Because $\mathrm{Re}(p+n)$ and $p+1-2\alpha$ need not vanish a priori, we argue near 0 first: ϖ is $A(z)$-analytic in a lemniscate $L(0,r)$ where $p \neq 2\alpha - 1$, and $\varpi(0) = 0$. Suppose the claim fails. Then there is a smallest $r_0 < 1$ such that $|\varpi| < 1$ in $L(0,r_0)$ and $|\varpi(z_0)| = 1$ at some $z_0 \in \partial L(0,r_0)$ (on $L(0,r_0)$ we have $\mathrm{Re}p > \alpha$, so ϖ is $A(z)$-analytic on a neighbourhood of $\overline{L(0,r_0)}$). Lemma 5.1 gives $\psi\ \partial_z\varpi = \kappa\varpi$ at z_0, $\kappa \geq 1$. Write $\varpi(z_0) = e^{i\theta}$, $\theta \neq 0$, and $y = \cot(\theta/2)$. Since ∂_z is a derivation,

$$p(z_0) = \alpha + i(1-\alpha)y,$$

$$\psi\ \partial_z p\,(z_0) = \frac{2(1-\alpha)\,\kappa\, e^{i\theta}}{(1-e^{i\theta})^2} = -\frac{(1-\alpha)\kappa(1+y^2)}{2} =: v \leq -\frac{(1-\alpha)(1+y^2)}{2}.$$

With $C = (\alpha+n)^2 + (1-\alpha)^2 y^2 > 0$,

$$\mathrm{Re}\left[p + \frac{\psi\ \partial_z p}{p+n}\right](z_0) = \alpha + \frac{(\alpha+n)\,v}{C} \leq$$

$$\leq \alpha - \frac{(1-\alpha)(\alpha+n)(1+y^2)}{2C} = \frac{\mathcal{P} + \mathcal{Q}\,y^2}{2C},$$

where $\mathcal{P} = (\alpha+n)[2\alpha(\alpha+n) - (1-\alpha)]$ and $\mathcal{Q} = (1-\alpha)[2\alpha(1-\alpha) - (\alpha+n)]$. The number α_n is the positive root of $2\alpha^2 + (2n+1)\alpha - 1 = 0$, hence $\mathcal{P} = 0$; using $2\alpha_n^2 = 1 - (2n+1)\alpha_n$ we find $2\alpha_n(1-\alpha_n) - (\alpha_n + n) = (n+1)(2\alpha_n - 1) \leq 0$, so $\mathcal{Q} \leq 0$. Thus the real part in (21) is ≤ 0 at z_0, a contradiction. Hence $|\varpi| < 1$ in $\mathcal{L}$, which is equivalent to $\mathrm{Re}p > \alpha_n$. ▫

Lemma 5.3 ($A(z)$-analogue of the Ruscheweyh-Sheil-Small theorem). *Let $\varphi \in C_A(0)$, $g \in S_A^\star(\alpha)$, $0 \leq \alpha < 1$, and $H \in \mathcal{O}_A(\mathcal{L})$. Then the values of $\frac{\varphi \star_A (Hg)}{\varphi \star_A g}$ lie in the closed convex hull of $H(\mathcal{L})$, and $\varphi \star_A g \in S_A^\star(\alpha)$.*

Proof. By (14), $\Phi = \varphi \circ \psi^{-1}$ is convex and $G = g \circ \psi^{-1} \in S^\star(\alpha) \subset S^\star$. Because $\varphi \star_A (Hg) = \Big(\Phi \star \big((H \circ \psi^{-1})G\big)\Big) \circ \psi$ and $\psi(\mathcal{L}) = E$, the first assertion is the theorem of Ruscheweyh and Sheil-Small [12]. Taking $H = \psi\ \partial_z g/g$, whose values lie in $\{\mathrm{Re}\xi > \alpha\}$, and using $\varphi \star_A (\psi\ \partial_z g) = \psi\ \partial_z(\varphi \star_A g)$, we obtain the second assertion. ▫

Lemma 5.4. *For $n \in \mathbb{N}_0$ and $f \in \mathcal{A}_A$ define*

$$\mathcal{L}_A(f)(z) = \frac{n+1}{\psi(z)^n} \int_{\gamma_z} \psi\,(t)^{n-1} f(t)\,\big(dt + A(t)\,d\bar{t}\big),$$

$$\gamma_z = \psi^{-1}([0, \psi(z)]). \quad (23)$$

Then $\mathcal{L}_A(f) = \psi + \sum_{m\geq 2} \frac{n+1}{m+n} a_m \psi^m$,

$$I_A^n(\mathcal{L}_A f) = \mathcal{L}_A(I_A^n f) = I_A^{n+1} f, \qquad (24)$$

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and $\mathcal{L}_A$ maps $S_A^\star(\alpha)$ into itself for $0 \le \alpha < 1$.

Proof. Along γ_z the substitution $u = \psi(t)$, $du = dt + A\,d\overleftarrow{t}$, turns (23) into the Bernardi integral $\frac{n+1}{w^n}\int_0^w u^{n-1}F(u)\,du$ with $w = \psi(z)$; by Theorem 2.1 the path γ_z may be replaced by any path from 0 to z in $\mathcal{L}$. Its coefficients are $\frac{n+1}{m+n}a_m$, and $c_m(n)\frac{n+1}{m+n} = c_m(n+1)$ gives (24), convolution operators being commutative. For the last claim let $g \in S_A^\star(\alpha)$ and $h = \mathcal{L}_A(g)$. Differentiating (23) we get $\psi\ \partial_z h + nh = (n+1)g$. Put $q = \psi\ \partial_z h/h$; then $(q+n)h = (n+1)g$ and $\frac{\psi\ \partial_z g}{g} = q + \frac{\psi\ \partial_z q}{q+n}$. If $\mathrm{Re}\,q \le \alpha$ somewhere, write $q = \alpha + (1-\alpha)\frac{1+\varpi}{1-\varpi}$ and apply Lemma 5.1 at the first point where $|\varpi| = 1$: as in the proof of Lemma 5.2 one gets $\mathrm{Re}\frac{\psi\ \partial_z g}{g} \le \alpha$ there, a contradiction. (Equivalently, this is Bernardi's theorem [1] transferred by (14).) ▫

6. Main results

6.1. Inclusion

Theorem 6.1. *For every $n \in \mathbb{N}_0$, $R_{2,A}^\star(n) \subset R_{2,A}^\star(n+1, \alpha_n)$, where α_n is given by* (22).

Proof. Let $f \in R_{2,A}^\star(n)$, so $I_A^n f \in S_A^\star$. By (24) and Lemma 5.4, $I_A^{n+1}f = \mathcal{L}_A(I_A^n f) \in S_A^\star$; hence $I_A^{n+1}f \ne 0$ in $\mathcal{L}\backslash\{0\}$ and

$$h = \frac{\psi\ \partial_z(I_A^{n+1}f)}{I_A^{n+1}f} \in \mathcal{O}_A(\mathcal{L}), \quad h(0) = 1.$$

Proposition 4.5(a) gives $(n+1)\,I_A^n f = (h+n)\,I_A^{n+1}f$. Applying the logarithmic $A(z)$-derivative $\psi\ \partial_z \log$ to both sides,

$$\frac{\psi\ \partial_z(I_A^n f)}{I_A^n f} = h + \frac{\psi\ \partial_z h}{h+n}.$$

The left side has positive real part, so by Lemma 5.2 $\mathrm{Re}\,h > \alpha_n$, i.e. $f \in R_{2,A}^\star(n+1, \alpha_n)$. ▫

Corollary 6.2. *(i) ($n = 0$) Every $A(z)$-convex function is $A(z)$-starlike of order $1/2$: if $\mathrm{Re}(1 + \psi\ \partial_z^2 f/\partial_z f) > 0$ in $\mathcal{L}$, then $\mathrm{Re}(\psi\ \partial_z f/f) > 1/2$ in $\mathcal{L}$. (ii) ($n = 1$) $S_A^\star \subset R_{2,A}^\star\big(2, (\sqrt{17}-3)/4\big)$.*

The constant $1/2$ in (i) is sharp: for $f = \psi/(1-\psi)$ we have $\psi\ \partial_z f/f = 1/(1-\psi)$, whose real part tends to $1/2$ as $\psi(z) \to -1$.

6.2. Convolution with $\boldsymbol{A(z)}$-convex functions

Theorem 6.3. *If $f \in R_{2,A}^\star(n, \alpha)$ and $\varphi \in C_A(0)$, then $f \star_A \varphi \in R_{2,A}^\star(n, \alpha)$.*

Proof. By the commutativity and associativity of $\star_A$, $I_A^n(f \star_A \varphi) = (I_A^n f) \star_A \varphi$. Since $I_A^n f \in S_A^\star(\alpha)$, Lemma 5.3 yields $(I_A^n f) \star_A \varphi \in S_A^\star(\alpha)$. ▫

Corollary 6.4. *Let $d\psi(t) = dt + A(t)\,d\overleftarrow{t}$. The classes $R_{2,A}^\star(n, \alpha)$ are invariant under the operators*

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$$f_1(z)=\int_{\gamma_z}\frac{f(t)}{\psi(t)}\,d\psi(t),\qquad f_2(z)=\frac{2}{\psi(z)}\int_{\gamma_z}f\,(t)\,d\psi(t),$$

$$f_3(z)=\int_{\gamma_z}\frac{f(t)-f\big(\omega_x(t)\big)}{(1-x)\,\psi(t)}\,d\psi(t)\ (|x|\le 1,\ x\ne 1),$$

$$f_4(z)=\frac{1+\beta}{\psi(z)^{\beta}}\int_{\gamma_z}\psi\,(t)^{\beta-1}f(t)\,d\psi(t)\ (\mathrm{Re}\beta>0).$$

Proof. In the variable $w=\psi(z)$ these are the Alexander, Libera, difference and Bernardi operators, so $f_i=f\star_A(\Phi_i\circ\psi)$ with

$$\Phi_1=-\log(1-w),\quad \Phi_2=-\frac{2\,[w+\log(1-w)]}{w},\quad \Phi_3=\frac{1}{1-x}\log\frac{1-xw}{1-w},\quad \Phi_4=\sum_{m\ge1}\frac{1+\beta}{m+\beta}\,w^m.$$

These functions are convex in E [11, 12], hence $\Phi_i\circ\psi\in C_A(0)$ by (14), and Theorem 6.3 applies. ▫

6.3. Coefficients and growth

Theorem 6.5. *Let $f\in R^{\star}_{k,A}(n,\alpha)$ be given by (13). Then the coefficients $a_m=\frac{1}{2\pi i}\int_{\partial L(0,\rho)}\frac{f(\zeta)\,(d\zeta+A(\zeta)\,d\bar{\zeta})}{\psi(\zeta)^{m+1}}$ satisfy:*

(a) for $k\ge 2$ and $m>3$, $a_m=O(1)\,m^{(1-\alpha)(k/2+1)+n-2}$, where $O(1)$ depends only on k,α,n, and the exponent cannot be decreased;

(b) for $k=2$ and $m\ge 2$,

$$|a_m|\le\frac{(m+n-1)!}{n!\;m!}\prod_{j=2}^{m}\frac{j-2\alpha}{j-1},\qquad(25)$$

with equality for the function determined by $I_A^n f=\psi(1-\psi)^{-2(1-\alpha)}$.

Proof. (b) Let $g=I_A^n f=\psi+\sum b_m\psi^m$, $b_m=c_m(n)a_m$, and $q=\psi\;\partial_z g/g\in P_A(\alpha)$. From $\psi\;\partial_z g=q\,g$ and $\psi\;\partial_z\psi^m=m\psi^m$, comparing coefficients of ψ^m (Theorem 3.3 makes this legitimate) gives $(m-1)b_m=\sum_{j=1}^{m-1}q_{m-j}\,b_j$, $b_1=1$, where $q=1+\sum q_j\psi^j$. By Proposition 4.2 applied to $(q-\alpha)/(1-\alpha)$, $|q_j|\le 2(1-\alpha)$. Induction yields $|b_m|\le\frac{2(1-\alpha)}{m-1}\sum_{j=1}^{m-1}|b_j|\le\prod_{j=2}^{m}\frac{j-2\alpha}{j-1}$, because the right-hand sides B_m satisfy $(m-1)B_m=2(1-\alpha)\sum_{j<m}B_j$. Dividing by $c_m(n)$ gives (25); for the indicated function all inequalities become equalities.

(a) By (20) the numbers $c_m(n)a_m$ are the coefficients of $I_nF\in R_k^{\star}(\alpha)$, and the estimate of Noor [5] gives $c_m(n)|a_m|\le C(k,\alpha)\,m^{(1-\alpha)(k/2+1)-1}$. Since $1/c_m(n)=\frac{(m+n-1)!}{n!\,m!}\sim\frac{m^{n-1}}{n!}$, the bound follows. The function of Example 4.7 has ψ-coefficients

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of exact order $m^{(1-\alpha)(k/2+1)+n-2}$ (when $\delta_1 \neq \delta_2$), which shows that the exponent is best possible. ▫

Theorem 6.6. *Let $f \in R^{\star}_{2,A}(n,\alpha)$ and $g = I^n_A f$. Then for $z \in \mathcal{L}$*

$$\frac{|\psi(z)|}{(1+|\psi(z)|)^{2(1-\alpha)}} \leq |g(z)| \leq \frac{|\psi(z)|}{(1-|\psi(z)|)^{2(1-\alpha)}}, \qquad (26)$$

and, for $|z| < (1+c)^{-1}$,

$$\frac{(1-c)|z|}{(1+(1+c)|z|)^{2(1-\alpha)}} \leq |g(z)| \leq \frac{(1+c)|z|}{(1-(1+c)|z|)^{2(1-\alpha)}}.$$

The estimates (26) are sharp.

Proof. Fix $z \in \mathcal{L}$, $\psi(z) = re^{i\theta}$, and integrate along the $A(z)$-radius γ_z. With $q = \psi\ \partial_z g/g$ and $s = \psi(t)$, $t \in \gamma_z$, we have $\frac{d}{ds}\log\frac{g}{\psi} = \frac{q-1}{s}$, hence

$$\log\left|\frac{g(z)}{\psi(z)}\right| = \int_0^r \mathrm{Re}\,(q(t)-1)\frac{d\rho}{\rho}, \qquad \psi(t) = \rho e^{i\theta}.$$

By Proposition 4.2, $\frac{q-\alpha}{1-\alpha}$ has the form (15), so $\frac{1-\rho}{1+\rho} \leq \frac{\mathrm{Re}q-\alpha}{1-\alpha} \leq \frac{1+\rho}{1-\rho}$ on $|\psi| = \rho$. Therefore $-\frac{2(1-\alpha)}{1+\rho} \leq \frac{\mathrm{Re}q-1}{\rho} \leq \frac{2(1-\alpha)}{1-\rho}$, and integration gives (26). Equality holds for $g = \psi(1-\delta\psi)^{-2(1-\alpha)}$, $|\delta| = 1$. The second pair of inequalities follows from (26), Lemma 3.1 and the monotonicity of $t(1 \mp t)^{-2(1-\alpha)}$ on $[0,1)$. ▫

6.4. The operator $\mathcal{L}_A$ and a radius problem

Theorem 6.7. *Let $k \geq 2$. If $f \in R^{\star}_{k,A}(n+1)$, then $\mathcal{L}_A(f) \in R^{\star}_{k,A}(n)$.*

Proof. By (24), $I^n_A(\mathcal{L}_A f) = I^{n+1}_A f \in R^{\star}_{k,A}(0)$. ▫

Theorem 6.8. *Let $\mathcal{L}_A(f) \in R^{\star}_{2,A}(n)$. Then $I^n_A f$ is $A(z)$-starlike in the lemniscate $L(0,r_n)$:*

$$\mathrm{Re}\frac{\psi(z)\ \partial_z\big(I^n_A f(z)\big)}{I^n_A f(z)} > 0 \quad \text{for } |\psi(z)| < r_n, r_n = \frac{1+n}{2+\sqrt{3+n^2}}. \qquad (27)$$

Proof. Put $g = I^{n+1}_A f = I^n_A(\mathcal{L}_A f) \in S^{\star}_A$ and $q = \psi\ \partial_z g/g \in P_A$. By Proposition 4.5 (a), $(n+1)I^n_A f = (q+n)g$, and therefore

$$\frac{\psi\ \partial_z(I^n_A f)}{I^n_A f} = q + \frac{\psi\ \partial_z q}{q+n}.$$

Let $|\psi(z)| = r < 1$. Proposition 4.2 gives the classical bounds $\mathrm{Re}q \geq \frac{1-r}{1+r}$ and $|\psi\ \partial_z q| \leq \frac{2r}{1-r^2}\mathrm{Re}q$ (differentiate (15) under the integral sign). Writing $x = \mathrm{Re}q$ and using $|q+n| \geq x+n$,

$$\mathrm{Re}\left[q + \frac{\psi\ \partial_z q}{q+n}\right] \geq x - \frac{2r}{1-r^2}\frac{x}{x+n} = \frac{x}{x+n}\left(x+n-\frac{2r}{1-r^2}\right).$$

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The last factor is increasing in x, and for $x = \frac{1-r}{1+r}$ it is positive exactly when $(1-r)^2 + n(1-r^2) > 2r$, i.e. $(n-1)r^2 + 4r - (n+1) < 0$; the positive root of this quadratic is $r_n = \frac{1+n}{2+\sqrt{3+n^2}}$. Hence the real part is positive for $|\psi(z)| < r_n$. ▫

Remark 6.9. In [8] the radius r_n appears as the radius of convexity of $\phi_n(w) = \frac{n}{n+1}\frac{w}{1-w} + \frac{1}{n+1}\frac{w}{(1-w)^2}$, for which $F = \phi_n \star \mathcal{L}(F)$. Theorem 6.8 obtains the same radius by a direct estimate in the lemniscate, without convolution arguments.

6.5. Recovery from a part of the boundary

Theorem 6.10. *Let $f \in R^{\star}_{k,A}(n,\alpha)$, $n \geq 1$, and suppose $g = I^n_A f \in H^1_A(\mathcal{L})$. Let $M \subset \partial\mathcal{L}$ have positive measure and let φ be the quenching function of M (Theorem 2.3). Put*

$$g_m(z) = \frac{1}{2\pi i}\int_M g^*(\zeta)\left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m \frac{d\zeta + A(\zeta)\,d\bar{\zeta}}{\psi(\zeta) - \psi(z)}.$$

Then, uniformly on compact subsets of $\mathcal{L}$,

$$f(z) = \lim_{m\to\infty} \frac{1}{n!}\,\partial_z^{n-1}\big(\psi^{n-1}(z)\,g_m(z)\big). \qquad (28)$$

For $n = 0$, $f(z) = \lim_{m\to\infty}\int_{\gamma_z} g_m(t)\,\frac{d\psi(t)}{\psi(t)}$.

Proof. By (6), $\psi(\zeta, z) = \psi(\zeta) - \psi(z)$, so Theorem 2.3 with $a = 0$, $r = 1$ gives $g_m \to g$ uniformly on compact subsets of $\mathcal{L}$; each g_m is $A(z)$-analytic in z. Let $K \subset \mathcal{L}$ be compact and $K \subset L(0,\rho)$, $\rho < \rho' < 1$. The functions $G_m = (\psi^{n-1}g_m)\circ\psi^{-1}$ are holomorphic and converge uniformly on $|w| \leq \rho'$; by the Cauchy estimates their derivatives of order $n-1$ converge uniformly on $|w| \leq \rho$. By (12), $\partial_z^{n-1}(\psi^{n-1}g_m) \to \partial_z^{n-1}(\psi^{n-1}g)$ uniformly on K, and (19) gives (28). For $n = 0$ we have $g = \psi\,\partial_z f$, hence $f(z) = \int_{\gamma_z} g\; d\psi/\psi$, and the limit can be taken under the integral. ▫

Remark 6.11. By (20), every statement about the classical classes $R^{\star}_k(n,\alpha)$ that is expressed through Taylor coefficients, convolutions, the operator $w\,d/dw$ and values in E has an exact counterpart for $R^{\star}_{k,A}(n,\alpha)$ in $\mathcal{L}$, and also in any lemniscate $L(a,r) \subset \subset D$ (Remark 3.4). In Theorems 6.1, 6.3 and 6.8 we have restricted ourselves to $k = 2$, for which complete proofs are given above by means of Lemmas 5.1-5.4.

7. Conclusion

The $A(z)$-lemniscate $L(0,1)$ together with the coordinate ψ provides a natural setting for geometric function theory of $A(z)$-analytic functions: the operator $\partial/\partial z$ acts as the derivative in ψ, the $A(z)$-convolution, the $A(z)$-Ruscheweyh derivative and the $A(z)$-Noor integral operator are well defined, and the classes $R^{\star}_{k,A}(n,\alpha)$ inherit the inclusion, convolution, coefficient and radius properties of their classical prototypes. In addition, the $A(z)$-Carleman formula makes it possible to reconstruct a function of these classes from the values of $I^n_A f$ on a part of the boundary of the lemniscate.

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