Convergence of sequences of A(Z)-analytic functions

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Abstract: We study sequences of $A(z)$-analytic functions, i.e. solutions of the Beltrami equation $f_{\bar{z}} = A(z)f_z$ with an antianalytic coefficient $A(z)$, $|A(z)| \le c < 1$, in a convex domain $D \subset \mathbb{C}$. Using the function $\psi(z,a)$ that generates the $A(z)$-lemniscates, we first prove that $\psi(\cdot,a)$ is a bi-Lipschitz diffeomorphism of D and obtain an explicit factorization $f = F \circ \psi(\cdot,a)$ of every $A(z)$-analytic function through a holomorphic function F. On this basis we prove analogues of two classical theorems: the Weierstrass theorem on uniformly convergent sequences (the limit is $A(z)$-analytic and the derivatives $\partial^k f_n / \partial z^k$ converge uniformly inside D) and the Hurwitz theorem on the number of a-points of the limit function. Each theorem is proved by two independent methods. The first method is intrinsic: it uses a Morera-type primitive, Cauchy inequalities in lemniscates and an argument principle for $A(z)$-analytic functions. The second method reduces the problem to holomorphic functions. As corollaries we obtain term-by-term differentiation of series, the preservation of univalence under limits, and a Vitali-type theorem.

Keywords: $A(z)$-analytic function, Beltrami equation, $A(z)$-lemniscate, uniform convergence inside a domain, Weierstrass theorem, Hurwitz theorem, argument principle

1. Introduction

Many arguments of geometric function theory rest on special convergence properties of sequences of holomorphic functions. The two main facts are the Weierstrass theorem and the Hurwitz theorem. By the Weierstrass theorem, a sequence of holomorphic functions converging uniformly on compact subsets has a holomorphic limit, and the derivatives of all orders converge as well. By the Hurwitz theorem, the limit function cannot have more a-points than the members of the sequence, unless it is constant (see, e.g., [38, 39]). These two facts are behind the proofs of the Riemann mapping theorem, the Montel and Vitali theorems and many extremal problems for univalent functions. It is therefore natural to ask whether they remain true for wider classes of functions that keep the main features of holomorphy.

Such a class is formed by the solutions of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = A(z)\,\frac{\partial f}{\partial z}, \qquad |A(z)| \le c < 1, \qquad (1.1)$$

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which are closely related to quasiconformal mappings [1-5]. For a measurable coefficient A, every generalized solution of (1.1) can be written as $f = \Phi \circ \chi$, where χ is a homeomorphic solution and Φ is holomorphic [2, 3]. In general, however, χ is not explicit, so this representation is of limited use for constructive purposes. A.L. Bukhgeim and S.G. Kazantsev [6] connected equations of the form (1.1) with problems of tomography, and this motivated a detailed study of the case when $A(z)$ is *antianalytic*.

For antianalytic $A(z)$, A.Sadullaev and N.M.Jabborov [7] constructed an explicit Cauchy kernel. They proved the integral theorem and the integral formula of Cauchy, the Taylor and Laurent expansions, the Montel theorem and the big Picard theorem for the class of $A(z)$-analytic functions. The boundary theory of $A(z)$-analytic functions was developed mainly by N.M. Zhabborov and B.E. Husenov and their co-authors. The Hardy class H_A^1 was introduced and studied in [8,15,20,22,31], and the Cauchy integral formula for functions of this class was obtained in [25,28]. The existence of angular limits for functions of the Nevanlinna class N_A was proved by A.Sadullaev and B.E.Husenov [9]. The Fatou theorem and the related radial and non-tangential boundary behaviour were studied in [10,24,32]. Boundary uniqueness theorems for bounded and Hardy-class $A(z)$-analytic functions were obtained in [11, 21, 23, 27, 33]. Analogues of the Goluzin-Krylov theorem, of the theorems of M.M. Lavrentiev and of the theorem of A.M. Kytmanov were obtained in [15-19]. The Smirnov and Riesz theorems were extended in [35, 36]. Carleman-type formulas, which reconstruct an $A(z)$-analytic function from its values on part of the boundary, together with the construction of quenching functions and applications, were given in [12-14,26,29,30]. The analytic-geometric properties of automorphisms of $A(z)$-lemniscates were studied in [34,37].

In the papers listed above, sequences of $A(z)$-analytic functions appear only implicitly: through the Montel theorem in [7] and through the Vitali theorem in [10]. The aim of the present paper is to state and prove the basic convergence theorems for this class systematically. Our main tool is the observation that, in a convex domain, the function $\psi(z, a)$ that defines the $A(z)$-lemniscates is a bi-Lipschitz diffeomorphism (Lemma 2.1). This gives an explicit factorization of $A(z)$-analytic functions through holomorphic ones (Lemma 2.2), an argument principle (Lemma 2.4) and Cauchy inequalities that hold uniformly in the centre of the lemniscate (Lemma 2.3).

The classical proofs of the Weierstrass and Hurwitz theorems use Cauchy's formula over circles, the Heine-Borel covering argument and Rouché's theorem. Here we give different proofs. For the Weierstrass-type theorem (Theorem 4.1), regularity of the limit is obtained from a Morera-type primitive, and convergence of derivatives from uniform Cauchy inequalities, with no covering argument. For the Hurwitz-type theorem (Theorem 5.1), Rouché's theorem is replaced by the continuity of the

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logarithmic integral. Each theorem also receives a second proof by reduction to the holomorphic case.

2. Preliminaries

Throughout the paper, $D \subset \mathbb{C}$ is a convex domain and $A(z)$ is an antianalytic function in D, i.e. $\partial A/\partial z = 0$, such that $|A(z)| \le c < 1$ for all $z \in D$. A function $f \in C^1(D)$ is called $A(z)$-*analytic* in D if it satisfies (1.1). The class of such functions is denoted by $O_A(D)$. Since A is infinitely smooth, $O_A(D) \subset C^\infty(D)$ [4, 7]. Constants, sums and products of $A(z)$-analytic functions are $A(z)$-analytic, because the operator $\overleftarrow{D}_A = \partial_{\bar z} - A\,\partial_z$ is a derivation. For the same reason, if $f \in O_A(D)$ and $f \ne 0$ in a subdomain, then $1/f$ is $A(z)$-analytic there.

We write

$$f'(z) := \frac{\partial f}{\partial z}(z), \qquad f^{(k)}(z) := \frac{\partial^k f}{\partial z^k}(z), \qquad d\psi := dz + A(z)\,d\bar z.$$

For $f \in O_A(D)$, equation (1.1) gives $df = f_z\,dz + f_{\bar z}\,d\bar z = f'(z)\,d\psi$. Hence

$$d(f\,d\psi) = df \wedge d\psi = f'\,d\psi \wedge d\psi = 0, \qquad (2.1)$$

i.e. the form $f\,d\psi$ is closed. This is the differential form of the Cauchy theorem [7]: $\oint_{\partial G} f\ (dz + A\,d\bar z) = 0$ for $f \in O_A(G) \cap C(\bar G)$.

Since $\overline{A(z)}$ is holomorphic in the convex domain D, for $a \in D$ the function

$$\psi(z,a) = z - a + \overline{\int_{[a,z]} \overline{A(\tau)}\,d\tau}, \qquad z \in D, \qquad (2.2)$$

is well defined, and the integral may be taken along any path in D. Moreover, $\partial\psi/\partial z = 1$ and $\partial\psi/\partial\bar z = A(z)$, so $\psi(\cdot,a) \in O_A(D)$ and $d\psi(z,a) = dz + A(z)\,d\bar z$ [7]. The open set

$$L(a,r) = \{z \in D:\ |\psi(z,a)| < r\}$$

is the $A(z)$-*lemniscate* with centre a and radius r. The Cauchy kernel is $K(\xi,z) = \frac{1}{2\pi i\,\psi(\xi,z)}$. For a subdomain G, $\bar G \subset D$, with piecewise smooth boundary and $f \in O_A(G) \cap C(\bar G)$, the Cauchy formula [7] reads

$$f(z) = \frac{1}{2\pi i}\oint_{\partial G} \frac{f(\xi)}{\psi(\xi,z)}\left(d\xi + A(\xi)\,d\bar\xi\right), \qquad z \in G. \qquad (2.3)$$

The following lemma refines the known fact [7] that $\psi(\cdot,a)$ has a unique simple zero.

Lemma 2.1. *For all $z,\zeta,a \in D$:*

(i) $\psi(z,a) - \psi(\zeta,a) = \psi(z,\zeta)$;

(ii) $(1-c)\,|z-\zeta| \le |\psi(z,\zeta)| \le (1+c)\,|z-\zeta|$;

(iii) the map $\psi_a := \psi(\cdot,a)$ is a C^∞-diffeomorphism of D onto the domain $\Omega_a = \psi_a(D)$ with Jacobian $1 - |A(z)|^2 > 0$; both ψ_a and ψ_a^{-1} are Lipschitz;

(iv) $D \cap U\left(a, \frac{r}{1+c}\right) \subset L(a,r) \subset U\left(a, \frac{r}{1-c}\right)$. *If* $\overleftarrow{U}\left(a, \frac{r}{1-c}\right) \subset D$, *then* ψ_a *maps* $L(a,r)$ *onto the disc* $U(0,r)$. *Consequently,* $\overleftarrow{L}(a,r) \subset D$ *is a closed Jordan domain with smooth boundary* $\partial L(a,r) = \psi_a^{-1}(\{|w| = r\})$, *and*

$$\int_{\partial L(a,r)} \left| d\xi + A(\xi)\, d\overleftarrow{\xi} \right| = 2\pi r. \qquad (2.4)$$

Proof. (i) By the path independence of the integral, $\psi(z,a) - \psi(\zeta,a) = z - \zeta + \overline{\int_{[\zeta,z]} \overleftarrow{A(\tau)}\, d\tau} = \psi(z,\zeta)$.

(ii) Since D is convex, $[\zeta,z] \subset D$ and $\left|\int_{[\zeta,z]} \overleftarrow{A(\tau)}\, d\tau\right| \le c\,|z-\zeta|$. Both inequalities follow from the triangle inequality.

(iii) By (i) and (ii), $|\psi_a(z) - \psi_a(\zeta)| = |\psi(z,\zeta)| \ge (1-c)|z-\zeta|$, so ψ_a is injective and ψ_a^{-1} is Lipschitz with constant $(1-c)^{-1}$. The Jacobian equals $|\psi_z|^2 - |\psi_{\overleftarrow{z}}|^2 = 1 - |A|^2 \ge 1 - c^2 > 0$. By the inverse function theorem, ψ_a is an open map and a C^∞-diffeomorphism onto its image Ω_a, which is a domain.

(iv) The inclusions follow from (ii) with $\zeta = a$. Let $\overleftarrow{U}\left(a, \frac{r}{(1-c)}\right) \subset D$ and $V = \psi_a\big(L(a,r)\big) \subset U(0,r)$. The set V is open and non-empty. It is also relatively closed in $U(0,r)$. Indeed, if $\psi_a(z_j) \to w$ with $|w| < r$ and $z_j \in L(a,r)$, then $z_j \in \overleftarrow{U}\left(a, \frac{r}{(1-c)}\right)$, which is compact in D. Passing to a subsequence, $z_j \to z \in D$, and then $|\psi(z,a)| = |w| < r$, so $w = \psi_a(z) \in V$. Since $U(0,r)$ is connected, $V = U(0,r)$. Finally, $d\psi(\xi,a) = dw$ on $\partial L(a,r)$, which gives (2.4). ▫

Lemma 2.2 (factorization). *Let* $a \in D$. *A function* f *belongs to* $O_A(D)$ *if and only if* $F := f \circ \psi_a^{-1}$ *is holomorphic in* Ω_a. *In this case*

$$f^{(k)} = F^{(k)} \circ \psi_a \quad (k = 0,1,2,\dots), \qquad (2.5)$$

and in particular $f^{(k)} \in O_A(D)$.

Proof. By Lemma 2.1(iii), f and F are smooth simultaneously. Put $w = \psi_a(z)$, so $w_z = 1$, $w_{\overleftarrow{z}} = A$, $\overleftarrow{w}_z = \overline{w_{\overleftarrow{z}}} = \overleftarrow{A}$ and $\overleftarrow{w}_{\overleftarrow{z}} = 1$. The chain rule gives

$$f_{\overleftarrow{z}} = F_w A + F_{\overleftarrow{w}}, \qquad f_z = F_w + F_{\overleftarrow{w}}\overleftarrow{A}, \qquad f_{\overleftarrow{z}} - A f_z = (1 - |A|^2) F_{\overleftarrow{w}}.$$

Since $1 - |A|^2 > 0$, we have $f \in O_A(D)$ if and only if $F_{\overleftarrow{w}} = 0$. In that case $f_z = F' \circ \psi_a$, and (2.5) follows by induction, because $F^{(k)}$ is holomorphic. ▫

Remark 2.1. That $f' \in O_A$ can also be checked directly. Differentiating $f_{\overleftarrow{z}} = A f_z$ with respect to z and using $A_z = 0$ gives $(f')_{\overleftarrow{z}} = A\,(f')_z$. Lemma 2.2 is an explicit version, for antianalytic A, of the Stoilow-Bojarski representation $f = \Phi \circ \chi$ [2, 3, 7]. Here χ is given by formula (2.2) and needs no existence theorem.

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Lemma 2.3 (Cauchy formula for derivatives and Cauchy inequalities). *Let G, $\tilde{G} \subset D$, be a domain with piecewise smooth boundary, and let $f \in O_A(G) \cap C(\tilde{G})$. Then for $z \in G$ and $k = 0,1,2, \ldots$*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\partial G} \frac{f(\xi)\left(d\xi + A(\xi)\, d\tilde{\xi}\right)}{\psi(\xi, z)^{k+1}}. \qquad (2.6)$$

If $\overline{L}(a, \rho') \subset D$, $f \in O_A(D)$, $0 < \rho < \rho'$ and $M(\rho') = \max_{\partial L(a,\rho')} |f|$, then

$$\left|f^{(k)}(z)\right| \le \frac{k!\ \rho'\, M(\rho')}{(\rho' - \rho)^{k+1}}, \quad z \in \overline{L}(a, \rho), \left|f^{(k)}(a)\right| \le \frac{k!\ M(\rho')}{\rho'^{\,k}}. \qquad (2.7)$$

Proof. By Lemma 2.1(i), $\psi(\xi, z) = \psi(\xi, a) - \psi(z, a)$ and $\partial_z \psi(z, a) = 1$. Hence $\partial_z^k\, \psi(\xi, z)^{-1} = k!\ \psi(\xi, z)^{-k-1}$. For z in a compact subset of G the denominator is bounded away from zero, so differentiation of (2.3) under the integral sign gives (2.6). Now take $G = L(a, \rho')$. For $\xi \in \partial G$ and $z \in \overline{L}(a, \rho)$ we have $|\psi(\xi, z)| \ge |\psi(\xi, a)| - |\psi(z, a)| \ge \rho' - \rho$, and the length of ∂G with respect to $|d\psi|$ is $2\pi\rho'$ by (2.4). This proves the first estimate. The second one is the case $z = a$, $\rho = 0$. ▫

At $z = a$, formula (2.6) shows that the Taylor coefficients of [7, Theorem 3.1] are $c_k = f^{(k)}(a)/k!$. Then (2.7) is the Cauchy inequality of [7], written in terms of derivatives.

Lemma 2.4 (zeros and the argument principle). *Let $f \in O_A(D)$, $f \not\equiv 0$.*

(a) The zeros of f are isolated. If $f(z_0) = 0$, there are an integer $m \ge 1$ (the order of the zero) and a function h, $A(z)$-analytic near z_0 with $h(z_0) \ne 0$, such that $f(z) = \psi(z, z_0)^m h(z)$. If the zeros of f accumulate at a point of D, then $f \equiv 0$ (uniqueness theorem).

(b) Let G, $\tilde{G} \subset D$, be a domain with piecewise smooth boundary such that $f \ne 0$ on ∂G. Then the number $N(f, G)$ of zeros of f in G, counted with multiplicity, equals

$$N(f, G) = \frac{1}{2\pi i} \oint_{\partial G} \frac{f'(z)}{f(z)}\, (dz + A(z)\, d\tilde{z}). \qquad (2.8)$$

Proof. (a) Let $F = f \circ \psi_{z_0}^{-1}$. By Lemma 2.2, F is holomorphic, $F(0) = 0$ and $F \not\equiv 0$. Hence $F(w) = w^m H(w)$ with $H(0) \ne 0$, and we may take $h = H \circ \psi_{z_0}$. Because ψ_{z_0} is a homeomorphism of D onto the domain Ω_{z_0}, accumulation points of zeros correspond to each other. The uniqueness theorem for F then gives the uniqueness theorem for f.

(b) We give an intrinsic proof. By (a), f has finitely many zeros $z_1, \ldots, z_s$ in G, of orders $m_1, \ldots, m_s$. By (2.1), the form $\omega = (f'/f)\, d\psi$ is closed outside the zeros, because $f'/f \in O_A$ there. Remove small disjoint lemniscates $L(z_j, \varepsilon)$ and apply Stokes' formula to ω on the remaining domain: $\oint_{\partial G} \omega = \sum_j \oint_{\partial L(z_j,\varepsilon)} \omega$. Near z_j we have $f = \psi(\cdot, z_j)^{m_j} h_j$ and $\partial_z \psi(z, z_j) = 1$, so

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$$\frac{f'}{f}\,d\psi = m_j\,\frac{d\psi(z,z_j)}{\psi(z,z_j)} + \frac{h_j'}{h_j}\,d\psi.$$

By Lemma 2.1 (iv), $\psi(\cdot,z_j)$ maps $\partial L(z_j,\varepsilon)$ onto the circle $|w|=\varepsilon$. Hence the first term contributes $2\pi i\,m_j$. The second term is bounded by $2\pi\varepsilon\max|h_j'/h_j| \to 0$. Letting $\varepsilon \to 0$ gives (2.8). ▫

3. Convergence of sequences: definitions and first properties

Let $\{f_n(z)\}_{n\geq 1}$ be single-valued functions on a set $E \subset D$.

Definition 3.1. The sequence $\{f_n\}$ *converges on* E if the numerical sequence $\{f_n(z_0)\}$ converges for each $z_0 \in E$. The function $f(z) = \lim_{n\to\infty} f_n(z)$ is then the *limit function*. The sequence *converges uniformly on* E to a finite function f if $\sup_{z\in E}|f_n(z) - f(z)| \to 0$ as $n \to \infty$. Equivalently (Cauchy criterion), for every $\varepsilon > 0$ there is N such that $|f_m(z) - f_n(z)| < \varepsilon$ for all $m,n > N$ and all $z \in E$. If the f_n are defined in D, the sequence *converges uniformly inside* D if it converges uniformly on every compact set $K \subset D$.

Uniform convergence inside D is weaker than uniform convergence in D. For example, $f_n(z) = \psi(z,a)^n$ converges to 0 uniformly inside $L(a,1)$ (when $\overline{L}(a,1) \subset D$), but not uniformly in $L(a,1)$.

In the $A(z)$-analytic setting it is natural to test convergence on lemniscates instead of discs.

Proposition 3.1. *A sequence* $\{f_n\}$ *converges uniformly inside* D *if and only if every point* $a \in D$ *is the centre of a lemniscate* $L(a,r_a)$ *with* $\overline{L}(a,r_a) \subset D$ *on which the sequence converges uniformly.*

Proof. Necessity: by Lemma 2.1(iv), $\overline{L}(a,r)$ is compact in D for $r < (1-c)\mathrm{dist}(a,\partial D)$. Sufficiency: a compact set $K \subset D$ is covered by the open lemniscates $L(a,r_a)$, $a \in K$, and hence by finitely many of them. Uniform convergence on each of these finitely many sets gives uniform convergence on K. ▫

Proposition 3.2. *If the functions* f_n *are continuous on* E *and converge uniformly on* E *to a finite function* f*, then* f *is continuous on* E*. In particular, the limit of a sequence of continuous functions converging uniformly inside* D *is continuous in* D*.*

Proof. We argue by contradiction, using sequences. Suppose that f is discontinuous at $z_0 \in E$. Then there are $\varepsilon_0 > 0$ and points $z_j \in E$, $z_j \to z_0$, with $|f(z_j) - f(z_0)| \geq \varepsilon_0$. Choose n with $\sup_E|f_n - f| < \varepsilon_0/3$. Then for all j

$$|f_n(z_j) - f_n(z_0)| \geq |f(z_j) - f(z_0)| - |f_n(z_j) - f(z_j)| - |f_n(z_0) - f(z_0)| > \frac{\varepsilon_0}{3},$$

which contradicts the continuity of f_n at z_0. ▫

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4. The Weierstrass theorem for $A(z)$-analytic functions

Theorem 4.1. *Let $f_n \in O_A(D)$, $n = 1,2, \ldots$, and let $\{f_n\}$ converge uniformly inside D to a finite function f. Then $f \in O_A(D)$, and for every $k = 1,2, \ldots$ the sequence of derivatives $f_n^{(k)} = \partial^k f_n / \partial z^k$ converges uniformly inside D to $f^{(k)}$.*

We give two proofs. The first works only with the class O_A itself. The second reduces the statement to the holomorphic case.

First proof (Morera-type primitive and Cauchy inequalities).

Step 1: f is $A(z)$-analytic. By Proposition 3.2, $f \in C(D)$. Fix $a \in D$ and a disc $U = U(a, \delta)$ with $\overline{U} \subset D$. For $z \in U$ put

$$\Phi_n(z) = \int_{[a,z]} f_n(\zeta)\left(d\zeta + A(\zeta)\, d\bar{\zeta}\right), \qquad \Phi(z) = \int_{[a,z]} f(\zeta)\left(d\zeta + A(\zeta)\, d\bar{\zeta}\right).$$

Since $\left|d\zeta + A\, d\bar{\zeta}\right| \le (1 + c)|d\zeta|$, we have $\Phi_n \to \Phi$ uniformly in U. Let $T \subset U$ be the closed triangle with vertices $a, z, z + h$. By (2.1) and Stokes' formula, $\oint_{\partial T} f_n\, d\psi = 0$, and since $f_n \to f$ uniformly on ∂T, also $\oint_{\partial T} f\, d\psi = 0$. Therefore

$$\Phi(z + h) - \Phi(z) = \int_{[z,z+h]} f(\zeta)\left(d\zeta + A(\zeta)\, d\bar{\zeta}\right) = f(z)\left(h + A(z)\bar{h}\right) + o(|h|), h \to 0,$$

because f and A are continuous. Thus Φ is differentiable, with $\Phi_z = f$ and $\Phi_{\bar{z}} = Af$. Its partial derivatives are continuous, so $\Phi \in C^1(U)$ and $\Phi_{\bar{z}} = A\,\Phi_z$, i.e. $\Phi \in O_A(U)$. By Lemma 2.2 (or Remark 2.1), $f = \Phi' \in O_A(U)$. Since $a \in D$ is arbitrary, $f \in O_A(D)$.

Step 2: convergence of derivatives. Let $K \subset D$ be compact and $\delta = \operatorname{dist}(K, \partial D)$ (we put $\delta = 1$ if $D = \mathbb{C}$). Set

$$\rho = \frac{(1 - c)\,\delta}{2}, \qquad K' = \left\{z : \operatorname{dist}(z, K) \le \frac{\delta}{2}\right\}.$$

Then K' is a compact subset of D. By Lemma 2.1(iv), $\overline{L}(a, \rho) \subset \overline{U}(a, \delta/2) \subset K'$ for every $a \in K$. The function $g_n = f_n - f$ belongs to $O_A(D)$ by Step 1, so the second inequality in (2.7) applied at each $a \in K$ gives

$$\sup_{a \in K} |f_n^{(k)}(a) - f^{(k)}(a)| \le \frac{k!}{\rho^k} \max_{z \in K'} |f_n(z) - f(z)| \to 0, \quad n \to \infty. \ \blacksquare$$

The lemniscates in Step 2 have a common radius ρ for all centres $a \in K$. This is possible because of the two-sided comparison of lemniscates with discs (Lemma 2.1), so no Heine-Borel covering argument is needed.

Second proof (reduction to holomorphic functions). Fix $a \in D$ and put $F_n = f_n \circ \psi_a^{-1}$ and $F = f \circ \psi_a^{-1}$. By Lemma 2.2, the F_n are holomorphic in Ω_a. If $Q \subset \Omega_a$ is compact, then $\psi_a^{-1}(Q) \subset D$ is compact, so $F_n \to F$ uniformly on Q. By the classical Weierstrass theorem [38, 39], F is holomorphic and $F_n^{(k)} \to F^{(k)}$ uniformly inside Ω_a.

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By Lemma 2.2, $f = F \circ \psi_a \in O_A(D)$. For a compact $K \subset D$, the set $\psi_a(K)$ is compact, and (2.5) gives

$$\sup_K \left|f_n^{(k)} - f^{(k)}\right| = \sup_{\psi_a(K)} \left|F_n^{(k)} - F^{(k)}\right| \to 0. \ \blacksquare$$

Remark 4.1 (a quantitative estimate). The first estimate in (2.7), applied to $f_n - f$, gives a lemniscate version of the classical estimate over circles. If $\overleftarrow{L}(a,\rho') \subset D$ and $|f_n - f| < \varepsilon$ on $\partial L(a,\rho')$, then for $z \in \overleftarrow{L}(a,\rho)$, $\rho < \rho'$,

$$|f_n(z) - f(z)| < \frac{\rho'}{\rho' - \rho}\,\varepsilon, \qquad \left|f_n^{(k)}(z) - f^{(k)}(z)\right| < \frac{k!\,\rho'}{(\rho' - \rho)^{k+1}}\,\varepsilon.$$

These estimates have the same form as in the holomorphic case, and they do not depend on the coefficient $A(z)$.

Remark 4.2 (convergence of all derivatives). Unlike the holomorphic case, $\partial f / \partial \tilde{z}$ does not vanish. It still converges, because $\partial_{\tilde{z}} f_n = A\, f_n{}' \to A\, f' = \partial_{\tilde{z}} f$. More generally, using $A_z = 0$, $\left(f^{(j)}\right)_{\tilde{z}} = A\, f^{(j+1)}$ and induction, every mixed derivative $\partial^{j+l} f / \partial z^j\, \partial \tilde{z}^l$ is a linear combination of $f^{(j)}, \dots, f^{(j+l)}$ with coefficients that are polynomials in $A, \partial_{\tilde{z}} A, \dots, \partial_{\tilde{z}}^{l-1} A$. Hence Theorem 4.1 implies that $f_n \to f$ in the topology of $C^\infty(D)$.

Corollary 4.1 (series). *Let $u_j \in O_A(D)$, and let the series $\sum_{j=1}^{\infty} u_j(z)$ converge uniformly inside D. Then its sum $S(z)$ is $A(z)$-analytic in D, and the series may be differentiated term by term any number of times with respect to z:*

$$S^{(k)}(z) = \sum_{j=1}^{\infty} u_j^{(k)}(z),$$

where the differentiated series also converge uniformly inside D.

For example, if $\overleftarrow{L}(a,R) \subset D$, the Taylor series $\sum_k c_k\, \psi(z,a)^k$ of [7] converges uniformly inside $L(a,R)$. Since $\partial_z \psi(z,a)^k = k\, \psi(z,a)^{k-1}$, Corollary 4.1 gives $f'(z) = \sum_{k \geq 1} k\ c_k\, \psi(z,a)^{k-1}$, in agreement with Lemma 2.3.

5. The Hurwitz theorem for *A*(*z*)-analytic functions

Theorem 5.1. *Let $f_n \in O_A(D)$, $n = 1,2,\dots$, converge uniformly inside D to a function $f \not\equiv \text{const}$. Let $a \in \mathbb{C}$ and $p \geq 0$ be an integer. If each function f_n takes the value a at no more than p points of D, then f also takes the value a at no more than p points of D.*

First proof (continuity of the logarithmic integral). By Theorem 4.1, $f \in O_A(D)$. Suppose that f takes the value a at $p+1$ distinct points $z_1, \dots, z_{p+1} \in D$. Since $f - a \in O_A(D)$ and $f - a \not\equiv 0$, these zeros are isolated (Lemma 2.4(a)). By Lemma 2.1(iv) we can choose radii $r_1, \dots, r_{p+1} > 0$ with the following properties:

• the closed lemniscates $\overleftarrow{L}_k = \overleftarrow{L}(z_k, r_k)$ lie in D and are pairwise disjoint;

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• $f(z) \neq a$ on $\bar{L}_k \backslash \{z_k\}$.

Let $\Gamma = \bigcup_k \partial L_k$ and $m = \min_\Gamma |f - a| > 0$. By Theorem 4.1, $f_n \to f$ and ${f_n}' \to f'$ uniformly on the compact set Γ. Hence there is n_0 such that $|f_n - a| \geq m/2$ on Γ for $n \geq n_0$, and

$$\left| \frac{{f_n}'}{f_n - a} - \frac{f'}{f - a} \right| \leq \frac{|{f_n}' - f'|\,|f - a| + |f'|\,|f_n - f|}{|f_n - a|\,|f - a|}$$

$$\leq \frac{2}{m^2} \left(\max_\Gamma |f - a| \max_\Gamma |{f_n}' - f'| + \max_\Gamma |f'| \max_\Gamma |f_n - f| \right) \to 0$$

uniformly on Γ. By (2.4), each curve ∂L_k has $|d\psi|$-length $2\pi r_k$. Therefore, by (2.8),

$$N(f_n - a, L_k) = \frac{1}{2\pi i} \oint_{\partial L_k} \frac{{f_n}'}{f_n - a}\, d\psi \;\to\; \frac{1}{2\pi i} \oint_{\partial L_k} \frac{f'}{f - a}\, d\psi = N(f - a, L_k) \geq 1.$$

The numbers $N(f_n - a, L_k)$ are integers, so $N(f_n - a, L_k) = N(f - a, L_k) \geq 1$ for all sufficiently large n. Thus, for such n, the function f_n takes the value a in each of the $p + 1$ disjoint lemniscates L_k, i.e. at no fewer than $p + 1$ points of D. This contradicts the hypothesis. ▫

Second proof (reduction). Fix $b \in D$ and let $F_n = f_n \circ \psi_b^{-1}$ and $F = f \circ \psi_b^{-1}$. As in the second proof of Theorem 4.1, the F_n are holomorphic and converge to F uniformly inside Ω_b. Moreover, $F \not\equiv \text{const}$. Since $\psi_b : D \to \Omega_b$ is a bijection, F_n takes the value a at exactly as many points of Ω_b as f_n does in D, and the same holds for F and f. The classical Hurwitz theorem [38, 39] gives the result. ▫

Remark 5.1 (Rouché-type theorem). The argument principle (2.8) also gives an analogue of Rouché's theorem. *If $f, g \in O_A(D)$, $\bar{G} \subset D$ and $|g| < |f|$ on ∂G, then* $N(f + g, G) = N(f, G)$. Indeed, the integer $N(f + tg, G)$, $0 \leq t \leq 1$, is given by (2.8) with an integrand that depends continuously on t, so it is constant. With this theorem, the classical proof [38] can be repeated word for word, with discs replaced by lemniscates. The first proof above avoids it and uses the convergence of the logarithmic integrals directly.

Remark 5.2. In the classical theorem the domain may contain the point ∞, and the whole extended plane has to be excluded. In our setting $D \subset \mathbb{C}$ is convex. If $D = \mathbb{C}$, then $A \equiv \text{const}$ by Liouville's theorem [7], $\psi(z, 0) = z + A\bar{z}$ is an affine bijection of $\mathbb{C}$, and Theorem 5.1 holds without exceptions.

6. Corollaries

Corollary 6.1 (preservation of univalence). *Let the functions $f_n \in O_A(D)$ be univalent (injective) in D and converge uniformly inside D to f. Then f is either univalent in D or constant.*

Proof. If $f \not\equiv \text{const}$, apply Theorem 5.1 with $p = 1$ to each value $a \in \mathbb{C}$. ▫

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Corollary 6.2 (zero-free limits). *If the functions $f_n \in O_A(D)$ have no zeros in D and converge uniformly inside D to f, then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in D$.*

Proof. This is the case $p = 0$, $a = 0$ of Theorem 5.1. If f is constant, it is either the zero constant or a non-zero constant. ▫

For example, the functions $f_n(z) = \exp(\psi(z,a)/n)$ are $A(z)$-analytic (by Lemma 2.2) and have no zeros. They converge to 1 uniformly inside D, in agreement with Corollary 6.2.

Corollary 6.3 (Vitali-type theorem). *Let $\{f_n\} \subset O_A(D)$ be uniformly bounded on every compact subset of D, and let $\lim_{n\to\infty} f_n(z)$ exist for every z in a set $E \subset D$ that has a limit point in D. Then $\{f_n\}$ converges uniformly inside D to a function $f \in O_A(D)$.*

Proof. By the Montel theorem for $A(z)$-analytic functions [7, Theorem 4.2], every subsequence of $\{f_n\}$ contains a further subsequence that converges uniformly inside D. By Theorem 4.1, its limit is $A(z)$-analytic. Any two such limits g_1, g_2 coincide on E. By the uniqueness theorem (Lemma 2.4(a)) applied to $g_1 - g_2$, they coincide in D. Suppose that $\{f_n\}$ did not converge uniformly on some compact set K to this common limit f. Then there would be $\varepsilon_0 > 0$ and a subsequence with $\max_K \left|f_{n_j} - f\right| \geq \varepsilon_0$, and this subsequence would contain no further subsequence converging to f on K, which is a contradiction. ▫

Corollary 6.3 is the form of the Vitali theorem used in the proof of the Fatou theorem for $A(z)$-analytic functions [10]. The results of this paper therefore give a complete justification of that step.

7. Conclusion

We have extended the basic convergence theory of holomorphic functions to the class $O_A(D)$ of $A(z)$-analytic functions with an antianalytic coefficient $A(z)$, $|A| \leq c < 1$, in a convex domain D. The main results are:

1. the function $\psi(z,a)$ is a bi-Lipschitz diffeomorphism, and every $f \in O_A(D)$ factors explicitly as $f = F \circ \psi(\cdot, a)$ with F holomorphic (Lemmas 2.1, 2.2);

2. the Cauchy formula for derivatives, Cauchy inequalities with constants that do not depend on the centre of the lemniscate, and the argument principle (Lemmas 2.3, 2.4);

3. the Weierstrass-type theorem (Theorem 4.1), with convergence of all derivatives and, in fact, convergence in $C^\infty(D)$ (Remark 4.2);

4. the Hurwitz-type theorem (Theorem 5.1);

5. its consequences: term-by-term differentiation of series, preservation of univalence and of the absence of zeros, and a Vitali-type theorem (Section 6).

Each main theorem has two proofs. The intrinsic proofs (Morera-type primitive, uniform Cauchy inequalities, convergence of logarithmic integrals) use only the structure of the class O_A, so they can be expected to carry over to settings where a

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global factorization is unavailable, for example to $A(z)$-analytic functions of several variables. The reduction proofs show how Lemma 2.2 transfers classical results to the class $O_A(D)$. These results can be applied to extremal problems for $A(z)$-univalent functions and to the study of normal families in Hardy-type classes of $A(z)$-analytic functions [8,12-14,20,31].

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