Characteristic properties of boundary values of functions of the classes Na

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Abstract: Let $A(z)$ be an antiholomorphic function in a convex domain $D \subset \mathbb{C}$ with $|A(z)| \leq c < 1$, and let $L(a,r) \Subset D$ be an $A(z)$-lemniscate. We describe the functions φ, given on a set $E \subset \partial L(a,r)$ of positive measure, that coincide almost everywhere on E with the boundary values of a function from the Nevanlinna-type classes N_A, H_A^p, N_A^q, M_A and N_{*A} of $A(z)$-analytic functions. In each case the criterion is the existence of a sequence of $A(z)$-polynomials $\sum_k c_k \psi^k(z,a)$ that converges to φ almost everywhere on E and whose boundary integrals obey the integral restriction that defines the class. The proofs do not rely on Runge-type approximation. They combine three tools: a uniformizing transfer principle based on the function $\psi(z,a)$, an explicit quantitative truncation of the expansion of a function in powers of $\psi(z,a)$, and an estimate of two-constants type written through the $A(z)$-harmonic measure. To pass from convergence inside the lemniscate to the boundary we use an outer multiplier together with weak-$*$ convergence, and to identify the classes we use the de la Vallée-Poussin criterion.

Keywords: $A(z)$-analytic function, $A(z)$-lemniscate, Nevanlinna class, Hardy class, Privalov class, Smirnov class, boundary values, $A(z)$-polynomials

1. Introduction

Let $D \subset \mathbb{C}$ be a domain and let $A(z)$ be a measurable function with $|A(z)| \leq c < 1$ in D. Continuous solutions of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = A(z)\,\frac{\partial f}{\partial z} \qquad (1.1)$$

are called $A(z)$-analytic functions. The general theory of (1.1) is that of quasiconformal mappings [1,2]. In [3] A.Sadullaev and N.M.Jabborov singled out the case when $A(z)$ is *antiholomorphic*. In this case the solutions keep many properties of holomorphic functions. There are analogues of the Cauchy theorem and the Cauchy integral formula [22,26], and expansions in Taylor and Laurent series in powers of a special function $\psi(z,a)$ [3]. There is a Poisson formula and a theory of $A(z)$-harmonic functions and $A(z)$-harmonic measures [8,9]. The Schwarz lemma [16] and the automorphisms of $A(z)$-lemniscates [33, 36] have also been studied.

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The boundary behaviour of $A(z)$-analytic functions has been investigated systematically in recent years. The Hardy classes H_A^p were introduced and studied in [4,17,19,29]. The existence of radial, angular and nontangential boundary values and analogues of Fatou's theorem were obtained in [4,6,21,23,30]. Analogues of the theorems of F. and M. Riesz and of V. I. Smirnov were proved in [34,35]. Boundary uniqueness theorems of Luzin-Privalov type were established in [7,18,20,25,31].

Another line of research concerns the recovery of an $A(z)$-analytic function from its values on a part of the boundary. Analogues of the Goluzin-Krylov theorem and of the methods of M.M.Lavrentiev and A.M.Kytmanov were obtained in [11-15]. Quenching functions and Carleman formulas were constructed in [8-10,24,27,28]. Applications of these formulas are given in [32].

The natural domains in this theory are the $A(z)$-lemniscates $L(a,r) = \{|\psi(z,a)| < r\}$. They play the role of disks.

The starting point of the present paper is the work of A. Sadullaev and B. E. Husenov [5]. There the Nevanlinna class N_A of $A(z)$-analytic functions in a lemniscate $L(a,r)$ was introduced. It was proved that every $f \in N_A$ is the ratio of two bounded $A(z)$-analytic functions, $f = f_1/f_2$, and hence has radial (angular) boundary values $f^*(\zeta)$ almost everywhere on $\partial L(a,r)$. This raises a natural question: which functions on a part of $\partial L(a,r)$ arise as such boundary values? The present paper answers this question for N_A and for its subclasses.

In the classical theory, the boundary values of functions of the Nevanlinna class N, the Hardy classes H^p, the Privalov classes N^q $(q > 1)$, the class M defined by the radial maximal function and the Smirnov class $N_* = N^+$ have a characteristic description on an arbitrary set E of positive measure on the boundary. A function φ on E is a boundary function of the class if and only if it is an almost everywhere limit on E of polynomials whose boundary integrals satisfy the corresponding restriction. For N and H^p this goes back to G. Ts. Tumarkin [38,39]. In the standard approach, necessity follows from Runge (Mergelyan) approximation of the dilations $f(\rho z)$. Sufficiency follows from a Khinchin-Ostrowski type theorem on sequences with uniformly bounded characteristic [37,38].

The aim of this paper is to obtain the analogous description for the classes N_A, H_A^p, N_A^q, M_A and N_{*A} of $A(z)$-analytic functions in a lemniscate, with antiholomorphic A. Our method is different. It is based on the following observations.

1. For convex D, the function $\Psi = \psi(\cdot, a)$ is a real-analytic bi-Lipschitz diffeomorphism of $\overline{L(a,r)}$ onto the closed disk $\{|w| \le r\}$. A function f is $A(z)$-analytic if and only if $f \circ \Psi^{-1}$ is holomorphic (Lemma 2.3). Through Ψ, the boundary measure $\left|d\zeta + A\, d\bar{\zeta}\right|$ becomes arc length. So Ψ transfers every Nevanlinna-type object

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to the disk without distortion. This also makes the expansion method of [3] quantitative.

2. Instead of Runge's theorem we use an explicit truncation of the ψ-expansion of the A-dilations of f, with a geometric error bound (Lemma 3.1). The approximating $A(z)$-polynomials are written down explicitly.

3. Instead of the Khinchin-Ostrowski theorem we prove a self-contained $A(z)$-version (Lemma 3.3). Interior convergence follows from a two-constants estimate written through the $A(z)$-harmonic measure of [8, 9]. The boundary values are identified by multiplying by a bounded outer function, which depends only on the part of the sequence off the given set, and then passing to weak-$*$ limits.

4. The classes N_A^q, M_A and N_{*A} are identified by the de la Vallée-Poussin criterion and by a pointwise lower semicontinuity property of the A-radial maximal function.

2. Preliminaries

2.1. $\boldsymbol{A(z)}$-analytic functions and the function $\boldsymbol{\psi}$

Throughout, $D \subset \mathbb{C}$ is a convex domain, $A(z)$ is antiholomorphic in D, and $|A(z)| \le c < 1$ for $z \in D$. We denote by $\mathcal{O}_A(U)$ the class of $A(z)$-analytic functions in an open set $U \subset D$. Since $A \in C^\infty(D)$, we have $\mathcal{O}_A(U) \subset C^\infty(U)$ [2,3]. The operator $\overleftarrow{D}_A = \partial_{\bar z} - A\,\partial_z$ is a derivation. Hence $\mathcal{O}_A(U)$ is an algebra.

For $z, \xi \in D$ we put

$$\psi(z,\xi) = z - \xi + \int_{[\xi,z]} A(\tau)\, d\bar\tau. \qquad (2.1)$$

Since $\overline{A}$ is holomorphic and D is convex, the integral does not depend on the path in D. Formula (2.1) is the function $z - \xi + \int \overline{A(\tau)}\, d\tau$ of [3,8]. It follows that

$$\frac{\partial\psi}{\partial z} = 1, \frac{\partial\psi}{\partial\bar z} = A(z), d\psi = dz + A(z)\, d\bar z, \psi(z,\xi) = \psi(z,a) - \psi(\xi,a). \quad (2.2)$$

In particular, $\psi(\cdot,\xi) \in \mathcal{O}_A(D)$. The set $L(a,r) = \{z \in D\colon |\psi(z,a)| < r\}$ is the $A(z)$-*lemniscate* with centre a and radius r. We fix $a \in D$ and $r > 0$ with $L(a,r) \Subset D$, and write

$$\Psi(z) = \psi(z,a), \quad \Gamma = \partial L(a,r), \quad \Gamma_\rho = \{z \in D\colon |\Psi(z)| = \rho\}, \quad \Delta_\rho = \{|w| < \rho\}, \quad \mathbb{T}_\rho = \partial\Delta_\rho.$$

On Γ we use the normalized boundary measure

$$d\sigma_A(\zeta) = \frac{1}{2\pi r}\left|d\zeta + A(\zeta)\, d\bar\zeta\right|, \quad \sigma_A(\Gamma) = 1. \qquad (2.3)$$

It is exactly the measure used in the definition of the Hardy classes in [4,8,9]. A function of the form

$$P(z) = \sum_{k=0}^{n} c_k\, \psi^k(z,a), \qquad c_k \in \mathbb{C},$$

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is called an $A(z)$-*polynomial*. By (2.2), the set of $A(z)$-polynomials does not depend on the choice of the centre a. All $A(z)$-polynomials belong to $\mathcal{O}_A(D)$.

Example 2.1. Let $D = \{|z| < 1\}$ and $A(z) = \bar{z}/2$. Then $\psi(z,0) = z + \bar{z}^2/4$. The lemniscates $L(0,r) = \{|z + \bar{z}^2/4| < r\}$ satisfy $L(0,r) \Subset D$ for $r < 1/2$, and the $A(z)$-polynomials are $\sum_k c_k\ (z + \bar{z}^2/4)^k$. They are not holomorphic in z.

2.2. The transfer principle

Lemma 2.2.

1. For all $z, z' \in D$,

$$(1-c)|z-z'| \le |\Psi(z) - \Psi(z')| \le (1+c)|z-z'|. \qquad (2.4)$$

2. Ψ is a real-analytic diffeomorphism of D onto the open set $\Psi(D)$ with Jacobian $1 - |A|^2 > 0$. For $0 < \rho \le r$ it maps $L(a,\rho)$ onto Δ_ρ and Γ_ρ onto $\mathbb{T}_\rho$. In particular, $\Gamma = \Gamma_r$ is a real-analytic Jordan curve.

3. The image of σ_A under Ψ is the normalized arc length $dm_r = |dw|/(2\pi r)$ on $\mathbb{T}_r$. Moreover $\frac{1-c}{2\pi r}|d\zeta| \le d\sigma_A \le \frac{1+c}{2\pi r}|d\zeta|$. So σ_A-null sets are the arc-length null sets.

Proof. (i) By (2.2), $\Psi(z) - \Psi(z') = z - z' + \int_{[z',z]} A\ d\bar{\tau}$, and the modulus of the integral does not exceed $c|z-z'|$.

(ii) The Jacobian of Ψ equals $|\Psi_z|^2 - |\Psi_{\bar{z}}|^2 = 1 - |A|^2$. Injectivity follows from (2.4), and Ψ is real-analytic because A is. By definition $\Psi\big(L(a,r)\big) \subset \Delta_r$. The set $\Psi\big(L(a,r)\big) = \Delta_r \cap \Psi(D)$ is open and nonempty. It is also closed in Δ_r. Indeed, if $\Psi(z_n) \to w \in \Delta_r$ with $z_n \in L(a,r)$, then (z_n) is a Cauchy sequence by (2.4). So $z_n \to z \in \overline{L(a,r)} \subset D$, $\Psi(z) = w$ and $z \in L(a,r)$. Since Δ_r is connected, $\Psi\big(L(a,r)\big) = \Delta_r$. Hence $\Psi\colon \overline{L(a,r)} \to \overline{\Delta_r}$ is a homeomorphism, and it maps Γ onto $\mathbb{T}_r$. The same argument works for every $\rho \le r$.

(iii) By (2.2), $dw = d\Psi = d\zeta + A\,d\bar{\zeta}$, and $(1-c)|d\zeta| \le \left|d\zeta + A\,d\bar{\zeta}\right| \le (1+c)|d\zeta|$. ▫

Lemma 2.3 (Transfer principle). Let $U \subset D$ be open, $f \in C^1(U)$, and $F = f \circ \Psi^{-1}$ on $\Psi(U)$. Then

$$\frac{\partial f}{\partial \bar{z}} - A\frac{\partial f}{\partial z} = (1-|A|^2)\left(\frac{\partial F}{\partial \bar{w}} \circ \Psi\right). \qquad (2.5)$$

Consequently, $f \in \mathcal{O}_A(U)$ if and only if F is holomorphic in $\Psi(U)$.

Proof. Let $w = \Psi(z)$. By (2.2),

$$f_{\bar{z}} = F_w\Psi_{\bar{z}} + F_{\bar{w}}\overline{\Psi_z} = AF_w + F_{\bar{w}}, \qquad f_z = F_w\Psi_z + F_{\bar{w}}\overline{\Psi_{\bar{z}}} = F_w + \bar{A}F_{\bar{w}}.$$

Hence $f_{\bar{z}} - Af_z = (1-|A|^2)F_{\bar{w}}$. ▫

Lemma 2.3 recovers the results of [3] at once. If $F(w) = \sum_k c_k\, w^k$ in Δ_r, then

$$f(z) = \sum_{k=0}^{\infty} c_k\ \psi^k(z,a),\ c_k = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\xi)\left(d\xi + A(\xi)\,d\bar{\xi}\right)}{\psi^{k+1}(\xi,a)}, 0 < \rho < r. \qquad (2.6)$$

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This holds because $d\Psi = d\xi + A\,d\bar{\xi}$. This is the Taylor expansion of [3]. Moreover, a function u is $A(z)$-harmonic (respectively, $A(z)$-subharmonic, i.e. satisfying the mean value inequality over lemniscates) in $L(a,r)$ if and only if $u \circ \Psi^{-1}$ is harmonic (respectively, subharmonic) in Δ_r. Indeed, by (2.2), Ψ maps every lemniscate $L(b,\rho) \subset L(a,r)$ onto the disk $\{|w - \Psi(b)| < \rho\}$, and the averages over $\partial L(b,\rho)$ with respect to $\left|d\zeta + A\,d\bar{\zeta}\right|$ become ordinary circle averages. The $A(z)$-Poisson kernel of [8, 9]

$$P_A(z,\zeta) = \frac{r^2 - |\psi(z,a)|^2}{|\psi(\zeta,z)|^2} = \frac{r^2 - |\Psi(z)|^2}{|\Psi(\zeta) - \Psi(z)|^2}\ z \in L(a,r),\ \zeta \in \Gamma, \qquad (2.7)$$

is the classical Poisson kernel of Δ_r at $\big(\Psi(z), \Psi(\zeta)\big)$. For a measurable set $F \subset \Gamma$ we call

$$\omega_A(z,F) = \int_F P_A\,(z,\zeta)\,d\sigma_A(\zeta), \qquad z \in L(a,r), \qquad (2.8)$$

the $A(z)$-*harmonic measure* of F. It differs only in sign from the function $\omega\big(z,F,L(a,r)\big)$ of [8, 9]. For $|\Psi(z)| = s < r$ we have the elementary bounds

$$\frac{r-s}{r+s} \le P_A(z,\zeta) \le \frac{r+s}{r-s}, \qquad \text{so} \qquad \omega_A(z,F) \ge \frac{r-s}{r+s}\,\sigma_A(F). \qquad (2.9)$$

2.3. $\boldsymbol{A(z)}$-dilations, $\boldsymbol{A(z)}$-radii and the classes

For a function f defined in $L(a,r)$ and $0 \le t < 1$, the $A(z)$-*dilation* of f is

$$f_t(z) = f\left(\Psi^{-1}\big(t\,\Psi(z)\big)\right), \qquad z \in \overline{L(a,r)}.$$

If $f \in \mathcal{O}_A\big(L(a,r)\big)$, then $f_t \in \mathcal{O}_A\big(L(a,r)\big) \cap C\left(\overline{L(a,r)}\right)$, and $f_t \circ \Psi^{-1} = F(t\ \cdot)$. For $\zeta \in \Gamma$ the curve $\gamma_\zeta = \{\Psi^{-1}\big(t\Psi(\zeta)\big)\colon\ 0 \le t \le 1\}$ is called the A(z)-*radius* ending at ζ. We put

$$f^*(\zeta) = \lim_{t\to 1^-} f_t(\zeta), \qquad M_A f(\zeta) = \sup_{0\le t<1} |f_t(\zeta)|,$$

the $A(z)$-*radial limit* and the $A(z)$-*radial maximal function* of f. The function $M_A f$ is a supremum of continuous functions of ζ. So it is lower semicontinuous and hence measurable. By Lemma 2.2 (iii), for every function $u \ge 0$,

$$\int_\Gamma u\,(f_t)\,d\sigma_A = \int_{\mathbb{T}_r} u\left(F(t\eta)\right) dm_r(\eta)$$

$$= \frac{1}{2\pi t r}\int_{\Gamma_{tr}} u\left(f(z)\right)|dz + A(z)\,d\bar{z}|. \qquad (2.10)$$

So the means below coincide with the means used in [4,8].

Definition 2.4. Let $f \in \mathcal{O}_A\big(L(a,r)\big)$. We say that

• $f \in N_A$ (the $A(z)$-Nevanlinna class [5]) if $\| f \|_{N_A} = \sup_{0\le t<1} \int_\Gamma \ln^+ |f_t|\,d\sigma_A < \infty$;

• $f \in H_A^p$, $0 < p < \infty$ (the $A(z)$-Hardy class [4]), if $\sup_{t<1} \int_\Gamma |f_t|^p\ d\sigma_A < \infty$;

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• $f \in N_A^q$, $1 < q < \infty$ (the $A(z)$-Privalov class), if $\sup_{t<1} \int_\Gamma (\ln^+|f_t|)^q \, d\sigma_A < \infty$;

• $f \in M_A$ if $\int_\Gamma \ln^+ M_A f \, d\sigma_A < \infty$;

• $f \in N_{*A}$ (the $A(z)$-Smirnov class) if $f \in N_A$ and the family $\{\ln^+|f_t|\}_{0 \le t<1}$ is uniformly integrable on (Γ, σ_A).

Uniform integrability of a family $\{u_\alpha\} \subset L^1(\sigma_A)$ means equi-absolute continuity of the integrals: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\int_e |u_\alpha| \, d\sigma_A < \varepsilon$ for all α whenever $\sigma_A(e) < \delta$. Since σ_A has no atoms, this condition implies $\sup_\alpha \int_\Gamma |u_\alpha| \, d\sigma_A < \infty$: cover Γ by finitely many arcs of measure less than δ.

By Lemma 2.3 and (2.10), f belongs to one of the classes above if and only if $F = f \circ \Psi^{-1}$ belongs to the corresponding classical class in Δ_r. Moreover, $f^* = F^* \circ \Psi$ and $M_A f = (MF) \circ \Psi$, where MF is the classical radial maximal function. Therefore the classical facts [37, 40, 41] give the following. Part (i) was proved in [5] by a different method: there it follows from the representation of f as the ratio of two bounded $A(z)$-analytic functions.

Proposition 2.5.

1. Every $f \in N_A$ has a finite $A(z)$-radial limit $f^*(\zeta)$ for σ_A-almost every $\zeta \in \Gamma$. If $f \not\equiv 0$, then $\ln|f^*| \in L^1(\sigma_A)$.

2. $\bigcup_{p>0} H_A^p \subset \bigcap_{q>1} N_A^q$ and $\bigcup_{q>1} N_A^q \subset M_A \subset N_{*A} \subset N_A$.

For the reader's convenience we indicate why (ii) holds; by the transfer principle it suffices to argue in Δ_r.

• The first inclusion follows from $(\ln^+ x)^q \le C_{p,q} \, x^p$ for $x \ge 0$.

• $N^q \subset M$ for $q > 1$. A bound on $\int (\ln^+|F(t\,\cdot)|)^q$ makes the family $\{\ln^+|F(t\,\cdot)|\}$ uniformly integrable, so F lies in the Smirnov class. Then $\ln^+|F| \le \mathcal{P}[\ln^+|F^*|]$ [40, 41], where $\mathcal{P}$ is the Poisson integral. Hence $\ln^+ MF$ is dominated by the Hardy-Littlewood maximal function of $\ln^+|F^*| \in L^q$, which is integrable.

• $M \subset N_*$: the integrable function $\ln^+ MF$ majorizes every $\ln^+|F(t\,\cdot)|$.

• $N_* \subset N$ holds by definition.

Remark 2.6. By (2.4), Ψ^{-1} maps every Stolz angle of Δ_r at $\eta \in \mathbb{T}_r$ into a region $\{z: |z-\zeta| < C\operatorname{dist}(z,\Gamma)\}$ at $\zeta = \Psi^{-1}(\eta)$, and conversely. Hence, for $f \in N_A$, the limit $f^*(\zeta)$ is also a nontangential limit for σ_A-almost every ζ. In particular, it coincides almost everywhere with the angular boundary values used in [4,6,8].

3. Three auxiliary lemmas

3.1. Quantitative expansion

Lemma 3.1 (Expansion lemma). Let $f \in \mathcal{O}_A\big(L(a,r)\big)$, $0 < t < 1$, and $tr < \rho < r$. Put $M(\rho) = \max_{\overline{L(a,\rho)}} |f|$ and $q = tr/\rho$. Let $S_n(f_t)$ be the n-th partial sum of the ψ-expansion of f_t:

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$$S_n(f_t)(z) = \sum_{k=0}^{n} c_k\ t^k\, \psi^k(z,a),$$

where c_k are the coefficients (2.6) of f. Then $S_n(f_t)$ is an $A(z)$-polynomial and

$$\sup_{z\in\overline{L(a,r)}} |f_t(z) - S_n(f_t)(z)| \le M(\rho)\,\frac{q^{\,n+1}}{1-q}. \qquad (3.1)$$

Proof. By (2.6) and Lemma 2.2(iii), $\int_{\Gamma_\rho} \left|d\xi + A\, d\tilde{\xi}\right| = 2\pi\rho$. This gives the Cauchy inequality $|c_k| \le M(\rho)\rho^{-k}$. Since $\Psi\left(\Psi^{-1}\left(t\Psi(z)\right)\right) = t\Psi(z)$, we have $f_t(z) = \sum_k c_k\, t^k \psi^k(z,a)$ for $z \in \overline{L(a,r)}$. On $\overline{L(a,r)}$ we have $|\psi(z,a)| \le r$, so the k-th term is bounded by $M(\rho)q^k$. Summing the tail of the geometric series gives (3.1).

Taking $t_\nu = 1 - \frac{1}{\nu+1}$, $\rho_\nu = \frac{(1+t_\nu)r}{2}$, and choosing n_ν so large that the right-hand side of (3.1) is less than $1/\nu$, we obtain explicit $A(z)$-polynomials

$$P_\nu = S_{n_\nu}\left(f_{t_\nu}\right), \qquad \sup_{\overline{L(a,r)}} \left|P_\nu - f_{t_\nu}\right| < \frac{1}{\nu}, \qquad \nu \in \mathbb{N}. \qquad (3.2)$$

We call (P_ν) the *canonical approximating sequence* of f.

3.2. A two-constants estimate

Lemma 3.2. Let $g \in \mathcal{O}_A\left(L(a,r)\right) \cap C\left(\overline{L(a,r)}\right)$ and $\int_\Gamma \ln^+ |g|\, d\sigma_A \le K$. Let $F \subset \Gamma$ be a measurable set with $|g| \le \varepsilon < 1$ on F. Then for $z \in L(a,r)$ with $|\psi(z,a)| = s$,

$$\ln|g(z)| \le \omega_A(z,F)\ln\varepsilon + \frac{r+s}{r-s}\,K \le \frac{r-s}{r+s}\,\sigma_A(F)\ln\varepsilon + \frac{r+s}{r-s}\,K. \qquad (3.3)$$

Proof. The function $G = g \circ \Psi^{-1}$ is holomorphic in Δ_r and continuous on $\overline{\Delta_r}$. So $\ln|G|$ is subharmonic in Δ_r and is majorized by its Poisson integral over $\mathbb{T}_r$. To see this, write the sub-mean value inequality on circles $\mathbb{T}_\rho$, $\rho < r$, and let $\rho \to r$, using Fatou's lemma; the functions $\ln|G(\rho\eta)|$ are uniformly bounded above. Transferring back with (2.7) and Lemma 2.2(iii),

$$\ln|g(z)| \le \int_\Gamma P_A\,(z,\zeta)\ln|g(\zeta)|\, d\sigma_A(\zeta) \le \ln\varepsilon \int_F P_A\ d\sigma_A + \int_{\Gamma\setminus F} P_A\ \ln^+|g|\, d\sigma_A.$$

Now use (2.8) and (2.9). The second inequality in (3.3) uses $\ln\varepsilon < 0$. ▫

3.3. An $\boldsymbol{A(z)}$-analogue of the Khinchin-Ostrowski theorem

We use the following standard fact. If holomorphic functions u_ν in Δ_r satisfy $|u_\nu| \le C$ and $u_\nu \to u$ locally uniformly, then $u_\nu^* \to u^*$ weak-$*$ in $L^\infty(\mathbb{T}_r)$. Indeed, the Fourier coefficients of u_ν^* with negative indices vanish, and those with nonnegative indices are r^k times the Taylor coefficients of u_ν. These converge to the corresponding coefficients of u^*. Trigonometric polynomials are dense in $L^1(\mathbb{T}_r)$, and the sequence is bounded in L^∞.

Lemma 3.3. Let $g_\nu \in \mathcal{O}_A\left(L(a,r)\right) \cap C\left(\overline{L(a,r)}\right)$ satisfy

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$$\int_{\Gamma} \ln^{+} |g_\nu| \, d\sigma_A \le K, \qquad \nu \in \mathbb{N}. \tag{3.4}$$

Let $E \subset \Gamma$ with $\sigma_A(E) > 0$, and suppose that $g_\nu \to \varphi$ σ_A-almost everywhere on E, where $\varphi: E \to \mathbb{C}$ is finite. Then:

1. g_ν converges uniformly on compact subsets of $L(a,r)$ to a function $f \in N_A$, and $\| f \|_{N_A} \le K$;

2. $f^* = \varphi$ σ_A-almost everywhere on E.

Proof. (a) By Egorov's theorem there is a measurable set $F \subset E$ with $\sigma_A(F) > 0$ on which $g_\nu \to \varphi$ uniformly. Let $\varepsilon \in (0,1)$. Then $|g_\nu - g_\mu| \le \varepsilon$ on F for $\nu, \mu \ge \nu_0(\varepsilon)$. Since $\ln^+|x-y| \le \ln^+|x| + \ln^+|y| + \ln 2$, we have $\int_\Gamma \ln^+ |g_\nu - g_\mu| \, d\sigma_A \le 2K + \ln 2$. Lemma 3.2, applied to $g = g_\nu - g_\mu$, gives for $z \in \overline{L(a,s)}$, $s < r$:

$$\ln |g_\nu(z) - g_\mu(z)| \le \frac{r-s}{r+s} \, \sigma_A(F) \ln \varepsilon + \frac{r+s}{r-s} \, (2K + \ln 2).$$

The right-hand side tends to $-\infty$ as $\varepsilon \to 0$. So (g_ν) is uniformly Cauchy on each $\overline{L(a,s)}$, and it converges locally uniformly to a function f. The limit is $A(z)$-analytic: by Lemma 2.3, the functions $g_\nu \circ \Psi^{-1}$ are holomorphic and converge locally uniformly.

Fix $t < 1$. The function $\ln^+|g_\nu \circ \Psi^{-1}|$ is subharmonic and continuous on $\overline{\Delta_r}$. So its circle means increase with the radius, and by (2.10), $\int_\Gamma \ln^+ |(g_\nu)_t| \, d\sigma_A \le \int_\Gamma \ln^+ |g_\nu| \, d\sigma_A \le K$. The sequence $(g_\nu)_t$ converges to f_t uniformly on Γ. Hence $\int_\Gamma \ln^+ |f_t| \, d\sigma_A \le K$ for every $t < 1$, that is, $f \in N_A$ and $\| f \|_{N_A} \le K$.

(b) Since φ is finite, $E = \bigcup_k \{\zeta \in E: \, |\varphi(\zeta)| \le k\}$. By Egorov's theorem and the regularity of σ_A, E is, up to a null set, a countable union of closed sets F with the following property: $|\varphi| \le K_0$ on F and $g_\nu \to \varphi$ uniformly on F. It is enough to prove $f^* = \varphi$ almost everywhere on such a set F.

We pass to the disk. Put $G_\nu = g_\nu \circ \Psi^{-1}$, $G = f \circ \Psi^{-1}$, $F' = \Psi(F) \subset \mathbb{T}_r$, $\phi = \varphi \circ \Psi^{-1}$. Let $h_\nu = \chi_{\mathbb{T}_r \setminus F'} \ln^+ |G_\nu|$, and consider the outer functions

$$H_\nu(w) = \exp\left(-\int_{\mathbb{T}_r} \frac{\eta + w}{\eta - w} \, h_\nu(\eta) \, dm_r(\eta) \right), \qquad w \in \Delta_r.$$

Then $|H_\nu| = e^{-\mathcal{P}[h_\nu]} \le 1$, where $\mathcal{P}$ denotes the Poisson integral in Δ_r, and $H_\nu(0) = e^{-\int h_\nu dm_r} \ge e^{-K}$. There is ν_0 such that $|G_\nu| \le K_0 + 1$ on F' for $\nu \ge \nu_0$. For such ν, subharmonicity gives

$$\ln |G_\nu| \le \mathcal{P}[\ln^+ |G_\nu|] = \mathcal{P}[h_\nu] + \mathcal{P}[\chi_{F'} \ln^+ |G_\nu|] \le \mathcal{P}[h_\nu] + \ln(K_0 + 1).$$

So the functions $U_\nu = G_\nu H_\nu$ satisfy $|U_\nu| \le K_0 + 1$ in Δ_r.

By Montel's theorem we may pass to a subsequence along which $H_\nu \to H$ locally uniformly. Then $|H| \le 1$ and $H(0) \ge e^{-K}$, so $H \not\equiv 0$. Also $U_\nu \to U = GH$, by part (a).

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By the fact stated before the lemma, $H_\nu^* \to H^*$ and $U_\nu^* \to U^*$ weak-$*$ in $L^\infty(\mathbb{T}_r)$. Since G_ν is continuous on $\overline{\Delta_r}$, we have $U_\nu^* = G_\nu H_\nu^*$. Let $\chi \in L^1(F')$. Then

$$\left|\int_{F'} (G_\nu - \phi)\, H_\nu^* \chi\, dm_r\right| \le \sup_{F'}|G_\nu - \phi| \;\|\, \chi \,\|_{L^1} \to 0, \int_{F'} \phi\, H_\nu^* \chi\, dm_r \to \int_{F'} \phi\, H^* \chi\, dm_r,$$

because $\phi\chi \in L^1$. On the other hand, $\int_{F'} U_\nu^* \chi\, dm_r \to \int_{F'} U^* \chi\, dm_r$. Since χ is arbitrary, $U^* = \phi H^*$ almost everywhere on F'.

Finally, $G \in N(\Delta_r)$ by part (a), and $H \in H^\infty(\Delta_r)$. So $U^* = G^* H^*$ almost everywhere. Since $H \not\equiv 0$ is bounded, $\ln|H^*| \in L^1(\mathbb{T}_r)$, and therefore $H^* \neq 0$ almost everywhere. Hence $G^* = \phi$ almost everywhere on F'. Transferring back by Lemma 2.2(iii), we get $f^* = \varphi$ σ_A-almost everywhere on F. ▫

Remark 3.4. In part (b) the multiplier H_ν depends only on the values of g_ν *off* the set F. This is what makes the weak-$*$ argument possible: on F the factor G_ν converges uniformly, while off F the growth of G_ν is absorbed by H_ν.

4. Main results

In this section $E \subset \Gamma$ is a measurable set with $\sigma_A(E) > 0$, and $\varphi: E \to \mathbb{C}$ is a finite measurable function.

Theorem 4.1 (Class N_A). There is a function $f \in N_A$ with $f^* = \varphi$ σ_A-almost everywhere on E if and only if there is a sequence of $A(z)$-polynomials (P_ν) such that

1. $P_\nu(\zeta) \to \varphi(\zeta)$ for σ_A-almost every $\zeta \in E$;
2. $\limsup_{\nu\to\infty} \int_\Gamma \ln^+ |P_\nu(\zeta)|\, d\sigma_A(\zeta) < \infty$.

Proof. Necessity. Let (P_ν) be the canonical approximating sequence (3.2) of f. For σ_A-almost every $\zeta \in E$,

$$|P_\nu(\zeta) - \varphi(\zeta)| \le \left|P_\nu(\zeta) - f_{t_\nu}(\zeta)\right| + \left|f_{t_\nu}(\zeta) - f^*(\zeta)\right| < \frac{1}{\nu} + \left|f_{t_\nu}(\zeta) - f^*(\zeta)\right| \to 0$$

by Proposition 2.5 (i). This proves (1). Moreover, $|P_\nu| \le \left|f_{t_\nu}\right| + 1$, so $\ln^+|P_\nu| \le \ln^+\left|f_{t_\nu}\right| + \ln 2$ and

$$\int_\Gamma \ln^+ |P_\nu|\, d\sigma_A \le \| f \|_{N_A} + \ln 2,$$

which proves (2).

Sufficiency. After discarding finitely many terms, (3.4) holds for $g_\nu = P_\nu$ with some $K < \infty$. Lemma 3.3 gives $f \in N_A$ with $f^* = \varphi$ almost everywhere on E. ▫

Theorem 4.2 (Classes H_A^p and N_A^q). Let $0 < p < \infty$ and $1 < q < \infty$. There is a function $f \in H_A^p$ (respectively, $f \in N_A^q$) with $f^* = \varphi$ σ_A-almost everywhere on E if and only if there is a sequence of $A(z)$-polynomials (P_ν) satisfying condition (1) of Theorem 4.1 and

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$$(2_p) \quad \limsup_{\nu\to\infty}\int_{\Gamma}|P_\nu|^p\,d\sigma_A$$

$$<\infty \qquad \left(\text{respectively, } (2^q) \quad \limsup_{\nu\to\infty}\int_{\Gamma}(\ln^+|P_\nu|)^q\,d\sigma_A<\infty\right).$$

Proof. Necessity. Take the canonical sequence (3.2). Condition (1) holds as in Theorem 4.1. Write $P_\nu = f_{t_\nu} + e_\nu$ with $\sup|e_\nu| < 1/\nu$. If $p \geq 1$, Minkowski's inequality gives $\| P_\nu \|_{L^p(\sigma_A)} \leq \| f_{t_\nu} \|_{L^p(\sigma_A)} + 1$. If $0 < p < 1$, the inequality $|x+y|^p \leq |x|^p + |y|^p$ gives $\int |P_\nu|^p\,d\sigma_A \leq \int \left|f_{t_\nu}\right|^p d\sigma_A + 1$. In both cases (2_p) follows from $f \in H_A^p$. For N_A^q we use $\ln^+|P_\nu| \leq \ln^+\left|f_{t_\nu}\right| + \ln 2$ and Minkowski's inequality in $L^q(\sigma_A)$.

Sufficiency. Condition (2_p) implies condition (2) of Theorem 4.1, because $\ln^+ x \leq x^p/p$. Condition (2^q) implies it by Hölder's inequality. By Lemma 3.3, $P_\nu \to f$ locally uniformly, $f \in N_A$, and $f^* = \varphi$ almost everywhere on E. The functions $|P_\nu \circ \Psi^{-1}|^p = e^{p\ln|P_\nu\circ\Psi^{-1}|}$ and $(\ln^+|P_\nu \circ \Psi^{-1}|)^q$ are convex nondecreasing functions of subharmonic functions. Hence they are subharmonic and continuous on $\overline{\Delta_r}$, and their circle means increase with the radius. By (2.10), for every $t < 1$,

$$\int_{\Gamma}|f_t|^p\,d\sigma_A = \lim_{\nu\to\infty}\int_{\Gamma}|(P_\nu)_t|^p\,d\sigma_A \leq \liminf_{\nu\to\infty}\int_{\Gamma}|P_\nu|^p\,d\sigma_A < \infty,$$

and similarly for $(\ln^+|f_t|)^q$. So $f \in H_A^p$ (respectively, $f \in N_A^q$). ▫

Theorem 4.3 (Class M_A). There is a function $f \in M_A$ with $f^* = \varphi$ σ_A-almost everywhere on E if and only if there is a sequence of $A(z)$-polynomials (P_ν) satisfying condition (1) of Theorem 4.1 and

$$(2_M) \qquad \limsup_{\nu\to\infty}\int_{\Gamma}\ln^+ M_A P_\nu(\zeta)\,d\sigma_A(\zeta) < \infty.$$

Proof. Necessity. Let (P_ν) be the canonical sequence. For $\zeta \in \Gamma$ and $0 \leq s < 1$,

$$\left(f_{t_\nu}\right)_s(\zeta) = f\left(\Psi^{-1}\left(st_\nu\Psi(\zeta)\right)\right), \text{ so } \quad |(P_\nu)_s(\zeta)| \leq \left|\left(f_{t_\nu}\right)_s(\zeta)\right| + \frac{1}{\nu} \leq M_A f(\zeta) + 1.$$

Taking the supremum over s gives $M_A P_\nu \leq M_A f + 1$. Hence, $\int \ln^+ M_A P_\nu\,d\sigma_A \leq \int \ln^+ M_A f\,d\sigma_A + \ln 2$.

Sufficiency. Since P_ν is continuous on $\overline{L(a,r)}$, we have $|P_\nu(\zeta)| = \lim_{s\to 1}|(P_\nu)_s(\zeta)| \leq M_A P_\nu(\zeta)$. So (2_M) implies condition (2) of Theorem 4.1. By Lemma 3.3, $P_\nu \to f$ locally uniformly, $f \in N_A$, and $f^* = \varphi$ almost everywhere on E. Fix $\zeta \in \Gamma$ and $s < 1$. Since $\Psi^{-1}\left(s\Psi(\zeta)\right) \in L(a,r)$,

$$|f_s(\zeta)| = \lim_{\nu\to\infty}|(P_\nu)_s(\zeta)| \leq \liminf_{\nu\to\infty} M_A P_\nu(\zeta).$$

Taking the supremum over s gives the pointwise inequality $M_A f \leq \liminf_\nu M_A P_\nu$. The function $\ln^+$ is continuous and nondecreasing, so Fatou's lemma yields

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$$\int_\Gamma \ln^+ M_A f \, d\sigma_A \le \liminf_{\nu\to\infty} \int_\Gamma \ln^+ M_A P_\nu \, d\sigma_A < \infty.$$

Hence $f \in M_A$. ▫

Theorem 4.4 (Class N_{*A}). There is a function $f \in N_{*A}$ with $f^* = \varphi$ σ_A-almost everywhere on E if and only if there is a sequence of $A(z)$-polynomials (P_ν) satisfying condition (1) of Theorem 4.1 and

(2_*) the integrals of the family $\{\ln^+|P_\nu|\}_{\nu\in\mathbb{N}}$ are equi-absolutely continuous on (Γ, σ_A).

Proof. Necessity. For the canonical sequence and every measurable $e \subset \Gamma$,

$$\int_e \ln^+ |P_\nu| \, d\sigma_A \le \int_e \ln^+ \left|f_{t_\nu}\right| d\sigma_A + \sigma_A(e)\ln 2.$$

Since $\{\ln^+|f_t|\}_{t<1}$ is uniformly integrable, (2_*) follows.

Sufficiency. By the remark after Definition 2.4, (2_*) implies condition (2) of Theorem 4.1. So Lemma 3.3 gives $f \in N_A$ with $P_\nu \to f$ locally uniformly and $f^* = \varphi$ almost everywhere on E. It remains to prove that $\{\ln^+|f_t|\}$ is uniformly integrable.

By the de la Vallée-Poussin criterion, a family in L^1 of a probability space is uniformly integrable if and only if there is a convex nondecreasing function $\Phi\colon [0,\infty) \to [0,\infty)$ with $\Phi(x)/x \to \infty$ as $x \to \infty$ such that the integrals $\int \Phi(|u|)$ are uniformly bounded on the family. Apply the criterion to $\{\ln^+|P_\nu|\}$ and obtain such a Φ with $\sup_\nu \int_\Gamma \Phi\,(\ln^+|P_\nu|)\, d\sigma_A = C_\Phi < \infty$. The function $\Phi \circ \ln^+|P_\nu \circ \Psi^{-1}|$ is subharmonic and continuous on $\overline{\Delta_r}$. As in the proof of Theorem 4.2, for every $t < 1$,

$$\int_\Gamma \Phi\,(\ln^+|f_t|)\, d\sigma_A = \lim_{\nu\to\infty} \int_\Gamma \Phi\,(\ln^+|(P_\nu)_t|)\, d\sigma_A \le C_\Phi.$$

By the same criterion, the family $\{\ln^+|f_t|\}_{t<1}$ is uniformly integrable. Hence $f \in N_{*A}$. ▫

Corollary 4.5. Let $\mathcal{X}$ be one of the classes N_A, H_A^p, N_A^q, M_A, N_{*A}. Let $f_\nu \in \mathcal{X}$ satisfy the condition defining $\mathcal{X}$ uniformly in ν. For N_A, H_A^p, N_A^q this means that the corresponding suprema are bounded independently of ν. For M_A it means that $\sup_\nu \int_\Gamma \ln^+ M_A f_\nu \, d\sigma_A < \infty$. For N_{*A} it means that the whole family $\{\ln^+|(f_\nu)_t|\}_{\nu,t}$ is uniformly integrable. If $f_\nu^* \to \varphi$ almost everywhere on E with φ finite, then φ coincides almost everywhere on E with the boundary values of a function of $\mathcal{X}$.

Proof. Fix ν. Since $(f_\nu)_t \to f_\nu^*$ almost everywhere on Γ as $t \to 1$, Egorov's theorem gives $t_\nu < 1$ and a set $e_\nu \subset \Gamma$ with $\sigma_A(e_\nu) < 2^{-\nu}$ such that $\left|(f_\nu)_{t_\nu} - f_\nu^*\right| < \frac{1}{(2\nu)}$ on $\Gamma\backslash e_\nu$. By Lemma 3.1 there is an $A(z)$-polynomial $Q_\nu = S_n\left((f_\nu)_{t_\nu}\right)$ with $\sup_{\overline{L(a,r)}}\left|Q_\nu - (f_\nu)_{t_\nu}\right| < \frac{1}{(2\nu)}$. Almost every $\zeta \in \Gamma$ lies in only finitely many of the sets e_ν (Borel-Cantelli lemma). Hence $|Q_\nu - f_\nu^*| < \frac{1}{\nu}$ eventually, and $Q_\nu \to \varphi$ almost

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everywhere on E. The estimates in the necessity parts of Theorems 4.1-4.4 use only $\sup\left|Q_\nu - (f_\nu)_{t_\nu}\right| \le 1$, so they are uniform in ν. Now apply the sufficiency parts. ▫

Remark 4.6. If $A(z) \equiv 0$, then $\psi(z,a) = z - a$ and the $A(z)$-lemniscates are disks. Theorems 4.1-4.4 then become the one-variable characterizations of the boundary values of the classes N, H^p, N^q, M, N_* (see [38, 39] for N and H^p), and the proofs above give new proofs of them. For $A(z) \not\equiv 0$ the approximating functions are not polynomials in z. The $A(z)$-polynomials are the natural replacement, and Lemma 3.1 produces them explicitly from the ψ-expansion (2.6).

Remark 4.7. Theorem 4.1 complements the Carleman-type formulas of [8-10]. Those formulas recover a function of H_A^1 from its boundary values on a set $M \subset \Gamma$ of positive measure. Theorems 4.1-4.4 describe exactly which data on M can occur as such boundary values. The uniqueness of the recovered function follows from Proposition 2.5 (i) and the boundary uniqueness theorems of [4,7].

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