Asymptotic estimates of two-variable integrals involving special cases of the Fox H-function

Dilshoda Isroil-qizi Akramova
akramova.shoda@mail.ru
Xurshida G'afur-qizi Narzulloyeva
narzullayevnaxurshida@gmail.com
Bukhara State University

Abstract: We study the asymptotic behaviour of two-variable integrals whose kernels are special cases of the Fox H-function (the exponential, the Wright and the Mittag-Leffler functions), as the parameters x, y tend to zero or to infinity. By means of the Mellin transform we prove: 1) a two-variable Watson-type theorem for the leading term, 2) an estimate of the rate of convergence, 3) a complete asymptotic expansion with a remainder estimate for rapidly decaying kernels, and 4) a uniform estimate for all $x, y > 0$. The results are confirmed by numerical computations.

Keywords: Fox H-function, Mellin transform, asymptotic estimate, Watson's lemma, Mittag-Leffler function, Wright function, two-variable integral

1. Introduction. The Fox H-function [2] is a general special function defined by a Mellin-Barnes type contour integral. Its special cases include elementary functions, the Bessel function, the Meijer G-function, the Mittag-Leffler and Wright functions, and many other special functions [1, 3]. The theory of integral transforms with an H-function in the kernel (H-transforms) is presented in detail in the monograph by A.A. Kilbas and M. Saigo [1]. Solutions of fractional differential equations, anomalous diffusion and related processes are often expressed through such functions [6, 7, 9]. When these solutions are written as two-variable integrals, it becomes important to know their behaviour for large or small values of the parameters.

For one-variable integrals the asymptotic methods (Watson's lemma, the Mellin convolution method) are well developed [4, 5]. Properties of H-transforms in the one-variable case are given in [1]. In the present paper these ideas are applied to two-variable integrals whose kernels are special cases of the H-function. The main difference is that the two parameters x and y may tend to zero or to infinity independently of each other, and the results must cover all four such regimes.

The paper is organised as follows. Section 2 contains the necessary definitions and facts, Section 3 states the problem, Section 4 contains the main results (Theorems 1-4) with proofs, and Section 5 presents numerical examples.

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2. Preliminaries. 2.1. The H-function. Let $0 \le m \le q$, $0 \le n \le p$ be integers, $a_i, b_j \in \mathbb{C}$ and $A_i, B_j > 0$. The Fox H-function is defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \,\middle|\, (a_i, A_i)_{1,p};\ (b_j, B_j)_{1,q}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s)\, z^{-s}\, ds \qquad (1)$$

where

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - A_j s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right) \prod_{j=n+1}^{p} \Gamma\left(a_j + A_j s\right)} \qquad (2)$$

and $\mathcal{L}$ is a suitable contour [1].

2.2. The Mellin transform. For a function $K\colon (0,\infty) \to \mathbb{C}$ the expression

$$K^*(s) = \int_0^\infty K(u)\, u^{s-1}\, du$$

is called its Mellin transform; we also write $\mathcal{M}[K](s)$. Equality (1) says that $H_{p,q}^{m,n}$ is given by the inverse Mellin transform of $\mathcal{H}_{p,q}^{m,n}$; therefore, for suitable parameters [1],

$$(H_{p,q}^{m,n})^*(s) = \mathcal{H}_{p,q}^{m,n}(s) \qquad (3)$$

in the strip of convergence. Below, formula (3) is used only in three special cases, and in each of them it is checked directly.

2.3. Special cases. The following three functions are special cases of the Fox H-function.

(K1) The exponential function. $e^{-u} = H_{0,1}^{1,0}[u \,|\, -;\ (0,1)]$, and its Mellin transform is

$$\mathcal{M}[e^{-u}](s) = \Gamma(s), \qquad \mathrm{Re}\, s > 0.$$

(K2) The Wright function. For $0 < \mu < 1$ and $\beta \in \mathbb{R}$,

$$\Phi_{\mu,\beta}(u) = \sum_{k=0}^{\infty} \frac{(-u)^k}{k!\ \Gamma(\beta - \mu k)} = H_{1,1}^{1,0}[u \,|\, (\beta, \mu);\ (0,1)],$$

$$\mathcal{M}\left[\Phi_{\mu,\beta}\right](s) = \frac{\Gamma(s)}{\Gamma(\beta + \mu s)}, \qquad \mathrm{Re}\, s > 0.$$

In particular, $\Phi_{1/2,\,1/2}(u) = \pi^{-1/2} e^{-u^2/4}$.

(K3) The Mittag-Leffler function. For $0 < \alpha < 2$ and $\beta \in \mathbb{R}$,

$$E_{\alpha,\beta}(-u) = \sum_{k=0}^{\infty} \frac{(-u)^k}{\Gamma(\alpha k + \beta)} = H_{1,2}^{1,1}[u \,|\, (0,1);\ (0,1), (1-\beta, \alpha)],$$

$$\mathcal{M}\left[E_{\alpha,\beta}(-u)\right](s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)}, \qquad 0 < \mathrm{Re}\, s < 1.$$

The correctness of these equalities is seen from (2). For example, in (K3) we have $m = n = 1$, $(a_1, A_1) = (0,1)$, $(b_1, B_1) = (0,1)$, $(b_2, B_2) = (1-\beta, \alpha)$, so that $\mathcal{H}_{1,2}^{1,1}(s) = \Gamma(s)\Gamma(1-s)/\Gamma(\beta - \alpha s)$, and the sum of the residues at the poles $s = -k$

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gives the series $\sum_k (-u)^k /\Gamma(\alpha k+\beta)$. In (K2) the residue of $\Gamma(s)$ at $s=-k$ equals $(-1)^k/k!$, and the value of $1/\Gamma(\beta+\mu s)$ there is $1/\Gamma(\beta-\mu k)$. We check the stated particular value of the Wright function: for odd k we have $1/\Gamma(1/2-k/2)=0$, and for $k=2l$ we have $\Gamma(1/2-l)=(-4)^l\, l!\,\sqrt{\pi}/(2l)!$, so the series becomes $\pi^{-1/2}\sum_l (-u^2/4)^l /l! = \pi^{-1/2}e^{-u^2/4}$. The same follows from the Mellin transform: $\int_0^\infty u^{s-1}\,\pi^{-1/2}e^{-u^2/4}du = 2^{s-1}\Gamma(s/2)/\sqrt{\pi}$, which by Legendre's duplication formula equals $\Gamma(s)/\Gamma\big((s+1)/2\big)$.

2.4. Asymptotic facts. Two known facts are used below.

(F1) If $0<\alpha<2$ and $\beta\in\mathbb{R}$, there is $C>0$ such that for all $u\geq 0$

$$\left|E_{\alpha,\beta}(-u)\right| \leq \frac{C}{1+u}.$$

This follows from the asymptotic expansion of the Mittag-Leffler function for large arguments [6, 7].

(F2) If $0<\mu<1$, there are $c, C>0$ such that for all $u\geq 0$

$$\left|\Phi_{\mu,\beta}(u)\right| \leq C\exp\big(-c\,u^{1/(1-\mu)}\big).$$

This is a consequence of Wright's result [8]; for $\Phi_{\mu,1-\mu}$ (the M-function) it is also given in [9].

Thus the kernels (K1), (K2) are rapidly decaying (that is, $|K(u)|\leq C_n(1+u)^{-n}$ for every n), whereas (K3) decays only like u^{-1}.

3. Statement of the problem. Let $a,b>0$ and

$$f(t,s)=t^{a-1}s^{b-1}\varphi(t,s), \qquad t,s>0,$$

where φ is a bounded measurable function on $(0,\infty)^2$. Let K_1, K_2 be one of the kernels (K1)-(K3). We consider the two-variable integral

$$I(x,y)=\int_0^\infty\int_0^\infty K_1(xt)\,K_2(ys)\,f(t,s)\,dt\,ds, \qquad x,y>0. \qquad (4)$$

The following problems are posed.

Problem 1. Find the leading term of $I(x,y)$ as $x\to x_0$, $y\to y_0$ ($x_0,y_0\in\{0,\infty\}$, that is, in the four regimes: $x,y\to\infty$; $x,y\to 0$; $x\to\infty, y\to 0$; $x\to 0, y\to\infty$).

Problem 2. Estimate the rate of convergence to the leading term.

Problem 3. For rapidly decaying K_1, K_2 and sufficiently smooth φ, construct a complete asymptotic expansion of $I(x,y)$ as $x,y\to\infty$ and estimate the remainder.

Problem 4. Find a uniform estimate of $|I(x,y)|$ of the form $x^{-a}y^{-b}$ valid for all $x,y>0$.

4. Main results. The factor $x^{-a}y^{-b}$ plays the key role below: the substitution $u=xt$, $v=ys$ in (4) gives

$$I(x,y)=x^{-a}y^{-b}\int_0^\infty\int_0^\infty K_1(u)K_2(v)\,u^{a-1}v^{b-1}\,\varphi\left(\frac{u}{x},\frac{v}{y}\right)du\,dv. \qquad (5)$$

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Absolute convergence of (4) follows from this equality and the hypotheses of the following theorem.

Theorem 1 (leading term). Let $K_1, K_2 : (0,\infty) \to \mathbb{C}$ be measurable functions such that $u^{a-1}K_1(u)$ and $v^{b-1}K_2(v)$ are absolutely integrable on $(0,\infty)$, and let φ be a bounded measurable function on $(0,\infty)^2$. For $x_0, y_0 \in \{0,\infty\}$ put $t_0 = 0$ if $x_0 = \infty$, $t_0 = \infty$ if $x_0 = 0$, and define s_0 in the same way.

If $\varphi(t,s) \to \varphi_*$ as $(t,s) \to (t_0,s_0)$, then as $x \to x_0$, $y \to y_0$

$$x^a y^b\, I(x,y) \to \varphi_*\, K_1^*(a)\, K_2^*(b). \qquad (6)$$

If for every $s > 0$ we have $\varphi(t,s) \to \varphi_1(s)$ as $t \to t_0$, then as $x \to x_0$ (with $y > 0$ fixed)

$$x^a\, I(x,y) \to K_1^*(a) \int_0^\infty K_2\,(ys)\, s^{b-1}\varphi_1(s)\, ds.$$

Proof. (a) Let $|\varphi| \le M$. The integrand in (5) is bounded in modulus by $M\,|K_1(u)|u^{a-1} \cdot |K_2(v)|v^{b-1}$, which is integrable on $(0,\infty)^2$ by hypothesis (Tonelli's theorem). Hence (4) converges absolutely and (5) holds. For fixed $u, v > 0$ we have $u/x \to t_0$ and $v/y \to s_0$ as $x \to x_0$, $y \to y_0$, so $\varphi(u/x, v/y) \to \varphi_*$. By Lebesgue's dominated convergence theorem the integral in (5) tends to $\varphi_* \int\int K_1(u)K_2(v)u^{a-1}v^{b-1}dudv = \varphi_* K_1^*(a)K_2^*(b)$.

Similarly, the substitution $u = xt$ gives

$$x^a I(x,y) = \int_0^\infty K_1\,(u)u^{a-1}\left[\int_0^\infty K_2\,(ys)s^{b-1}\varphi\left(\frac{u}{x}, s\right)ds\right]du.$$

Since $|\varphi| \le M$ and $\int |K_2(ys)|s^{b-1}ds = y^{-b}\int |K_2(v)|v^{b-1}dv < \infty$, the inner integral tends to $\int K_2(ys)s^{b-1}\varphi_1(s)ds$ by dominated convergence and is uniformly bounded by $My^{-b} \parallel v^{b-1}K_2 \parallel_{L^1}$. For the outer integral the majorant is proportional to $|K_1(u)|u^{a-1}$, so dominated convergence applies again. ▫

Corollary 1 (special kernels). For the kernels (K1)-(K3) the values $K^*(a)$ follow from (3), and the hypotheses of Theorem 1(a) hold in the following cases:

Kernel K	$K^*(a)$	Condition on a
e^{-u}	$\Gamma(a)$	$a > 0$
$\Phi_{\mu,\beta}(u)$, $0 < \mu < 1$	$\dfrac{\Gamma(a)}{\Gamma(\beta + \mu a)}$	$a > 0$
$E_{\alpha,\beta}(-u)$, $0 < \alpha < 2$	$\dfrac{\pi}{\sin(\pi a)\,\Gamma(\beta - \alpha a)}$	$0 < a < 1$

The hypotheses follow from (F1), (F2): the kernels (K1), (K2) are bounded near $u = 0$ and decay rapidly at infinity, so $a > 0$ suffices for the integrability of $u^{a-1}K(u)$; for (K3) the bound is of order u^{a-1} near 0 and u^{a-2} at infinity, which gives integrability for $0 < a < 1$. In the last row of the table we used $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$. For example, if both kernels are of type (K3), $x, y \to \infty$ and $\varphi(t,s) \to \varphi_0$ as $t, s \to 0$, then

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$$I(x,y) \sim \varphi_0 \frac{\pi^2}{\sin(\pi a)\sin(\pi b)} \cdot \frac{x^{-a}y^{-b}}{\Gamma(\beta_1 - \alpha_1 a)\,\Gamma(\beta_2 - \alpha_2 b)}.$$

Remark 1. If $\varphi \equiv 1$, then (5) gives the exact equality $I(x,y) = K_1^*(a)K_2^*(b)\,x^{-a}y^{-b}$. Hence the order $x^{-a}y^{-b}$ cannot be improved. If $K^*(a) = 0$ (for instance, $1/\Gamma(\beta + \mu a) = 0$), the limit in (6) is zero and the main contribution comes from the subsequent terms; this case is covered by Theorem 3.

Theorem 2 (rate of convergence). Let the hypotheses of Theorem 1(a) hold, let φ_0 be a constant and let $L > 0$, $\delta > 0$ be such that for all $t, s > 0$

$$|\varphi(t,s) - \varphi_0| \le L\left(t^\delta + s^\delta\right). \qquad (6')$$

Denote $\mathcal{N}_i(\sigma) = \int_0^\infty |K_i(u)|\; u^{\sigma-1}du$ and assume $\mathcal{N}_1(a+\delta) < \infty$, $\mathcal{N}_2(b+\delta) < \infty$. Then for all $x, y > 0$

$$|x^a y^b I(x,y) - \varphi_0 K_1^*(a)K_2^*(b)|$$
$$\le L\left[\mathcal{N}_1(a+\delta)\,\mathcal{N}_2(b)\,x^{-\delta} + \mathcal{N}_1(a)\,\mathcal{N}_2(b+\delta)\,y^{-\delta}\right]. \qquad (6'')$$

Proof. From (5) and $\varphi_0 K_1^*(a)K_2^*(b) = \varphi_0 \int\int K_1 K_2 u^{a-1}v^{b-1}dudv$ we obtain

$$x^a y^b I(x,y) - \varphi_0 K_1^*(a)K_2^*(b) =$$
$$= \int_0^\infty \int_0^\infty K_1(u)K_2(v)\,u^{a-1}v^{b-1}\left[\varphi\left(\frac{u}{x},\frac{v}{y}\right) - \varphi_0\right]du\,dv.$$

By $(6')$ the modulus of the expression in brackets does not exceed $L\left((u/x)^\delta + (v/y)^\delta\right)$. Hence the modulus of the left-hand side is bounded by $L\,x^{-\delta}\mathcal{N}_1(a+\delta)\mathcal{N}_2(b) + L\,y^{-\delta}\mathcal{N}_1(a)\mathcal{N}_2(b+\delta)$. ▫

Remark 2. For the kernels (K1), (K2) we have $\mathcal{N}(\sigma) < \infty$ for all $\sigma > 0$, whereas for (K3) only for $0 < \sigma < 1$. Therefore, if a (K3) kernel is present, one needs $\delta < 1 - b$ (or $\delta < 1 - a$), and the rate of convergence in the corresponding variable is limited to $y^{-\delta}$ ($\delta < 1 - b$). This explains the slow convergence observed in Example 1 of Section 5: there $b = 1/2$ and $|\varphi - 1| \le \min(1, t+s) \le t^\delta + s^\delta$ $(0 < \delta \le 1)$, so the rate is $O\left(y^{-\delta}\right)$ for every $\delta < 1/2$.

Theorem 1 requires only boundedness of φ and the existence of a limit at the corner point. For smoother φ and rapidly decaying kernels a complete expansion can be obtained.

Theorem 3 (complete asymptotic expansion). Let $K_1, K_2 \colon (0,\infty) \to \mathbb{C}$ be continuous and rapidly decaying, that is, $|K_i(u)| \le C_{i,n}(1+u)^{-n}$ for every $n \in \mathbb{N}$ (in particular, K_i is bounded). Let $a, b > 0$, $\varphi \in C^N([0,\infty)^2)$, and let all derivatives of φ of order N have polynomial growth: $\left|\partial_t^j \partial_s^k \varphi(t,s)\right| \le M(1+t+s)^m$, $j + k = N$. Then as $\min(x,y) \to \infty$

$$I(x,y) = x^{-a}y^{-b}\left[\sum_{j+k<N} c_{jk}\ K_1^*(a+j)\,K_2^*(b+k)\,x^{-j}y^{-k} + O(\min(x,y)^{-N})\right], \qquad (7)$$

where $c_{jk} = \frac{1}{j!\,k!}\,\partial_t^j\,\partial_s^k\varphi(0,0)$.

Proof. We write φ by Taylor's formula with the integral remainder at $(0,0)$:

$$\varphi(t,s) = \sum_{j+k<N} c_{jk}\ t^j s^k + R_N(t,s),$$

$$R_N(t,s) = \sum_{j+k=N} \frac{N}{j!\ k!}\,t^j s^k \int_0^1 (1-\theta)^{N-1}\ \partial_t^j\,\partial_s^k\varphi(\theta t,\theta s)\,d\theta.$$

The polynomial bound for the derivatives and $\int_0^1(1-\theta)^{N-1}\,d\theta = 1/N$ give

$$|R_N(t,s)| \le M(1+t+s)^m \sum_{j+k=N} \frac{t^j s^k}{j!\ k!} = \frac{M}{N!}\,(1+t+s)^m\,(t+s)^N. \qquad (8)$$

It follows that φ itself has polynomial growth, so (4) converges absolutely. Substituting the terms of the Taylor polynomial into (5) we get

$$\int_0^\infty \int_0^\infty K_1(u)K_2(v)u^{a-1}v^{b-1}\left(\frac{u}{x}\right)^j\left(\frac{v}{y}\right)^k du\,dv = K_1^*(a+j)\,K_2^*(b+k)\,x^{-j}y^{-k};$$

these integrals converge because the kernels decay rapidly and $a, b > 0$.

For the remainder, for $x, y \ge 1$ we have $u/x + v/y \le (u+v)/\min(x,y)$ and $1 + u/x + v/y \le 1 + u + v$, so by (8)

$$\left|\int_0^\infty \int_0^\infty K_1\,K_2\,u^{a-1}v^{b-1}R_N\left(\frac{u}{x},\frac{v}{y}\right)du dv\right| \le \frac{M}{N!\ \min(x,y)^N}\int_0^\infty \int_0^\infty |K_1(u)K_2(v)|\ u^{a-1}v^{b-1}(u+v)^N(1+u+v)^m\,du\,dv.$$

The last integral is finite, since K_1, K_2 decay faster than any power and u^{a-1}, v^{b-1} are integrable as $u, v \to 0$. Hence the remainder is $O(\min(x,y)^{-N})$. ▫

Corollary 2. For the kernels (K1) and (K2) the coefficients in (7) are given by the values of $K^*(a+j)$ from (3):

$$K_1^*(a+j) = \frac{\Gamma(a+j)}{\Gamma\big(\beta_1 + \mu_1(a+j)\big)}, \qquad K_2^*(b+k) = \frac{\Gamma(b+k)}{\Gamma\big(\beta_2 + \mu_2(b+k)\big)}.$$

If the kernel is e^{-u}, the corresponding factor becomes $\Gamma(a+j)$ (this corresponds to $\mu \to 0$, $\beta = 1$), and (7) yields the two-variable Watson lemma.

Corollary 3. Taking $N = 2$ in Theorem 3, for $\varphi \in C^2([0,\infty)^2)$ we obtain

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$$I(x,y)=\frac{\varphi(0,0)\,K_1^*(a)K_2^*(b)}{x^a y^b}+\frac{\varphi_t(0,0)\,K_1^*(a+1)K_2^*(b)}{x^{a+1}y^b}$$
$$+\frac{\varphi_s(0,0)\,K_1^*(a)K_2^*(b+1)}{x^a y^{b+1}}+O\left(\frac{1}{x^a y^b \min(x,y)^2}\right),$$

where $\varphi_t=\partial_t\varphi$, $\varphi_s=\partial_s\varphi$. For the kernel (K2) the ratio $K^*(a+1)/K^*(a)$ equals

$$\frac{\Phi^*_{\mu,\beta}(a+1)}{\Phi^*_{\mu,\beta}(a)}=a\,\frac{\Gamma(\beta+\mu a)}{\Gamma(\beta+\mu a+\mu)}$$

(when $K^*(a)\neq 0$), and for the exponential it equals a. For instance, for $\mu=\beta=1/2$, $a=1$ the ratio is $2/\sqrt{\pi}$, which gives the term $-2/\left(\sqrt{\pi}\,y\right)$ appearing in Example 2 of Section 5.

For the Mittag-Leffler kernel an expansion of type (7) is not directly valid, because $E_{\alpha,\beta}(-u)$ decays only like u^{-1} and the moments $K^*(a+j)$ do not exist for $a+j\geq 1$. In this case a uniform estimate is still available.

Theorem 4 (uniform estimate). Let $|K_1(u)|\leq C_1(1+u)^{-\rho_1}$, $|K_2(v)|\leq C_2(1+v)^{-\rho_2}$, $0<a<\rho_1$, $0<b<\rho_2$ and $|\varphi|\leq M$. Then for all $x,y>0$

$$|I(x,y)|\leq M\,C_1C_2\,B(a,\rho_1-a)\,B(b,\rho_2-b)\;x^{-a}y^{-b},\qquad (9)$$

where B is Euler's beta function. In particular, for (K3) kernels ($\rho_1=\rho_2=1$) by (F1)

$$|I(x,y)|\leq M\,C_1C_2\,\frac{\pi^2}{\sin(\pi a)\sin(\pi b)}\;x^{-a}y^{-b},\qquad 0<a,b<1.$$

Proof. From (5) with $|\varphi|\leq M$,

$$|I(x,y)|\leq M\,x^{-a}y^{-b}\int_0^\infty |K_1(u)|\,u^{a-1}du\int_0^\infty |K_2(v)|\,v^{b-1}dv.$$

By hypothesis $\int_0^\infty |K_1(u)|\,u^{a-1}du\leq C_1\int_0^\infty u^{a-1}(1+u)^{-\rho_1}du=C_1B(a,\rho_1-a)$, and similarly for the second integral. In the last statement we used $B(a,1-a)=\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a)$. ▫

Remark 3. Theorems 1 and 4 together show that for the kernels (K1)-(K3) the order of $I(x,y)$ is bounded by $x^{-a}y^{-b}$ uniformly for all $x,y>0$, and in the limiting regimes this bound turns into formula (6) with the exact constant; that is, the estimate is sharp in order (Remark 1). Part (b) of Theorem 1 also covers the case when one of x, y stays fixed.

Remark 4 (necessity of the limit condition). The condition on the limit of φ at the corner point in Theorem 1 cannot be dropped. Indeed, $\varphi(t,s)=\sin(\ln t)$ is bounded but has no limit as $t\to 0$. Let $K_1(u)=e^{-u}$ and let K_2 be an arbitrary real-valued function with $v^{b-1}K_2\in L^1$. Since $\varphi(u/x,v/y)=\sin(\ln u-\ln x)=\mathrm{Im}\left(u^i x^{-i}\right)$ in (5), we get

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$$x^a y^b I(x,y) = K_2^*(b)\,\mathrm{Im}\left(x^{-i}\,\Gamma(a+i)\right)$$
$$= K_2^*(b)\,|\Gamma(a+i)|\,\sin(\arg\Gamma(a+i) - \ln x).$$

The function Γ has no zeros, so for $K_2^*(b) \neq 0$ this expression has no limit as $x \to \infty$: it oscillates periodically in $\ln x$.

5. Numerical examples. The computations below were carried out by quadrature (the double integral was computed after scaling the variables and mapping the infinite interval) and confirm the theoretical results.

Example 1 (Mittag-Leffler kernel, Theorems 1 and 2). Let $K_1(u) = e^{-u}$, $K_2(v) = E_{1/2,1}(-v) = e^{v^2}\mathrm{erfc}(v)$, $a = b = 1/2$, $\varphi(t,s) = \frac{1}{1+t+s}$:

$$I(x,y) = \int_0^\infty \int_0^\infty \frac{e^{-xt}\,E_{1/2,1}(-ys)}{\sqrt{ts}\,(1+t+s)}\,dt\,ds.$$

Here $\varphi_0 = 1$, $K_1^*(1/2) = \Gamma(1/2) = \sqrt{\pi}$, $K_2^*(1/2) = \Gamma(1/2)^2/\Gamma(3/4)$. By (6)

$$\sqrt{xy}\;I(x,y) \to \frac{\pi^{3/2}}{\Gamma(3/4)} = 4.5440\ldots \qquad (x,y \to \infty).$$

Table 1.

Computed values of $\sqrt{xy}\,I(x,y)$

x	y	$\sqrt{xy}\,I(x,y)$	$\sqrt{xy}\,I(x,y)/\left(\pi^{3/2}/\Gamma(3/4)\right)$
10	10	3.5488	0.7810
10^2	10^2	4.2249	0.9298
10^3	10^3	4.4440	0.9780
10^4	10^4	4.5125	0.9931
10^2	10^4	4.4906	0.9882

The ratio tends to one, in agreement with (6). The convergence is slow (apparently of order $\min(x,y)^{-1/2}$), because the Mittag-Leffler kernel decays only algebraically.

Example 2 (Wright kernel, Theorem 3). Let $K_1(u) = e^{-u}$, $K_2(v) = \Phi_{1/2,1/2}(v) = \pi^{-1/2}e^{-v^2/4}$, $a = b = 1$, $\varphi(t,s) = \frac{1}{1+t+s}$:

$$I(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty \int_0^\infty \frac{e^{-xt-y^2s^2/4}}{1+t+s}\,dt\,ds.$$

Here $c_{jk} = (-1)^{j+k}\binom{j+k}{j}$, $K_1^*(1+j) = j!$, $K_2^*(1+k) = 2^k\,\Gamma(k+1/2)/\sqrt{\pi}$, so by (7)

$$I(x,y) = \frac{1}{xy}\left[1 - \frac{1}{x} - \frac{2}{\sqrt{\pi}\,y} + \cdots\right] \qquad (x,y \to \infty),$$

and the general term is $(-1)^{j+k}\binom{j+k}{j}\,j!\;2^k\,\Gamma(k+1/2)\pi^{-1/2}\,x^{-1-j}y^{-1-k}$.

Table 2.

Relative error $|I - S_N|/I$ of the N-term sum of expansion (7) ($j + k < N$)

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$x = y$	$I(x, y)$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
20	$2.2671 \cdot 10^{-3}$	$1.03 \cdot 10^{-1}$	$1.46 \cdot 10^{-2}$	$2.63 \cdot 10^{-3}$	$5.76 \cdot 10^{-4}$
50	$3.8391 \cdot 10^{-4}$	$4.19 \cdot 10^{-2}$	$2.43 \cdot 10^{-3}$	$1.78 \cdot 10^{-4}$	$1.59 \cdot 10^{-5}$

As N grows the error decreases (for $x = y = 50$ by more than a factor of 10 at each step), which agrees with the remainder estimate $O(\min(x, y)^{-N})$ in (7).

Example 3 (mixed regimes, Theorem 1). Let again $K_1(u) = e^{-u}$, $K_2(v) = \Phi_{1/2,\,1/2}(v) = \pi^{-1/2} e^{-v^2/4}$ and $a = b = 1$; then $K_1^*(1) = \Gamma(1) = 1$, $K_2^*(1) = \Gamma(1)/\Gamma(1) = 1$. Consider two functions φ:

$$\varphi_1(t, s) = \frac{s}{(1 + t)(1 + s)}, \qquad \varphi_2(t, s) = \frac{t\,s}{(1 + t)(1 + s)}.$$

For φ_1 we have $\varphi_1 \to 1$ as $t \to 0$, $s \to \infty$, that is, the regime $x \to \infty$, $y \to 0$; for φ_2 we have $\varphi_2 \to 1$ as $t, s \to \infty$, that is, the regime $x, y \to 0$. By Theorem 1, in both cases

$$x\,y\,I(x, y) \to 1.$$

Table 3.

Values of $x\,y\,I(x, y)$ (mixed regime and the regime of small arguments)

φ_1: $(x,\ y)$	$xy\,I$	φ_2: $(x,\ y)$	$xy\,I$
$(10;\ 0.1)$	0.7716	$(0.1;\ 0.1)$	0.6729
$(10^2;\ 10^{-2})$	0.9622	$(10^{-2};\ 10^{-2})$	0.9321
$(10^3;\ 10^{-3})$	0.9949	$(10^{-3};\ 10^{-3})$	0.9896

The values converge to one: this confirms numerically the mixed regime ($x \to \infty$, $y \to 0$) and the regime in which both parameters tend to zero in Theorem 1.

We remark that an asymptotic method for double integrals involving Fox H-functions and their special cases, including Wright and Mittag-Leffler functions, may also be relevant to the spectral analysis of discrete Schrödinger-type operator matrices [10-15]. In particular, the Birman-Schwinger principle and the Faddeev-type reduction transform certain two- and three-particle spectral problems into integral operator equations whose kernels depend on the spectral parameter and quasi-momenta. The asymptotic estimates, convergence rates, and uniform bounds established in this work may provide useful tools for investigating the behavior of these integral kernels near the thresholds of the essential spectrum and, consequently, the asymptotic distribution of discrete eigenvalues [16-20].

6. Conclusion. For two-variable integrals whose kernels are special cases of the Fox H-function (the exponential, Wright and Mittag-Leffler functions) four kinds of results were obtained: a general expression for the leading term in all four regimes (Theorem 1), a sharp estimate of the rate of convergence (Theorem 2), a complete asymptotic expansion with an estimated remainder for rapidly decaying kernels (Theorem 3), and a uniform estimate for all $x, y > 0$ (Theorem 4). All constants are written explicitly through the Mellin transform (2) of the H-function. Generalisations

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to oscillating kernels such as the Bessel function (where the integral does not converge absolutely), to general $H_{p,q}^{m,n}$ kernels and to two-variable H-functions may be directions of further research.

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