A Ky Fan theorem's application in the theory of graph energy

Zabihullah Zahir
Parwan University, Afghanistan
Farzana Mohammadi
Balkh University, Afghanistan

Abstract: The energy of a graph G is equal to the total of its absolute eigenvalues, which is equal to the sum of its singular adjacency matrix values. Let x, y and z be matrices such that $x+y=z$. The Ky-Fan theorem proves an inequality between the sum of the singular values of z and the sum of the singular values of x and y. Several new inequalities as well as fresh proofs of several previously known inequalities are produced when this theorem is applied to the notion of graph energy.

Keywords: graph energy, invariants energy, Laplacian energy, Ky Fan theorem and inequality

Introduction

Simple graphs are the focus of the essay. Let $G=(V,E)$ be such a graph with a vertex.

Set $V=V(G)$ and the edge set $E=E(G)$. When the order and size of G are n and m, respectively, i.e, $|V|=n$ and $|E|=m$, we say that G is a (n,m)- graph.

Let $A=A(G)$ represent the $(0,1)$- adjacency matrix of G. The spectrum of the graph $G[2]$ is made up of its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

If $V_i \in V(G)$, then d_i denotes the vertex's degree with $i=1,2,\dots n$. $D=D(G)$ represents the (i,i)-entry of the diagonal matrix of order n.

Then

$$L=L(G)=D(G)-A(G) \tag{1}$$

The Laplacian matrix of G is designated as (1).

The Laplacian spectrum of the graph G is formed by its eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ $[7,8,17]$.

We will also require a second Laplacian-type matrix in the subsequent discussion, which is defined as

$$L^{\dagger}=L^{\dagger}(G):=D(G)+A(G) \tag{2}$$

The definition of the graph G's energy is

$$E = E(G) := \sum_{i=1}^{n} |\lambda_i|.$$

when contrasted with its Laplacian energy

$$LE = LE(G) := \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \quad (3)$$

The graph invariants energy and Laplacian energy are currently the subject of extensive research. A common A current bibliography, which only includes papers published in 2001 and later, can be found online at http://www.sgt.pep.ufrj.br and currently includes about 150 references. For more information on graph energy theory See the review [9] and the book [12]. The most recent papers [14, 21–22].

Provide an overview of the fundamentals of Laplacian energy.

Let I_n be the order n unit matrix. It will be important to note for the considerations that follow that, instead, via Eq. (3), the Laplacian energy can also be expressed as

$$LE(G) = \sum_{i=1}^{n} |\gamma_i| \quad (4)$$

Where $\gamma_i, i = 1, 2, ..., n$ are the eigenvalues of the matrix $L(G) - (2m/n) I_n$.

I. The Ky-Fan theorem

Assume that M is a square matrix of real and symmetric order n. Consider the singular values of $s_i(M)$,where $i = 1, 2, ..., n$ as well as its eigenvalues, For is then $s_i(M) = |x_i(m)|$ for $i = 1, 2, ..., n$ The energy of the graph G is known by Nikiforov [18] to be the sum of the singular values of its adjacency matrix $A(G)$. The following theorem, which Fan first established [5], suggests that this observation is extremely significant for the theory of graph energy.

Theorem 1 [5]. Assume that $x + y = z$ and Let x, y and z be square matrices of order n. Then $\sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y) \geq \sum_{i=1}^{n} s_i(Z)$

Equality holds if and only if there exists an orthogonal matrix P where PX and PY are both positive semi-definite.

You can read more about the Ky Fan theorem in [3, 4] and the references cited there.

II. Some basic applications of the Ky Fan theorem

Since Theorem 1 states that for graphs G_x, G_y and G_z whose adjacency matrices meet the requirement $A(G_x) + A(G_y) = A(G_z)$.

$$E(G_x) + E(G_y) \geq E(G_z).$$

The following corollaries list some particular examples of this inequality:

Corollary 2. Assume that G is an order n graph, and that $\overline{G}$ stands for it's complement. Then

$$E(G)+E(\overline{G})\geq 2n-1$$

If and only if $G\cong K_n$ or $G\cong\overline{K}_n$, equality is maintained.

Proof. By noting that $A(G)+A(\overline{G})=A(K_n)$ and $E(K_n)=2(n-1)$ the inequality follows. Assume that the eigenvalues of G are for the purpose of establishing the requirements for equality.

$$\lambda_1(G)\geq\lambda_2(G)\geq\lambda_n(G)$$

Then

$$E(G)+E(\overline{G})\geq 2\lambda_1(G)+2\lambda(\overline{G}).$$

$$\geq 2\frac{2m(G)}{n}+\frac{2m(\overline{G})}{n}=\frac{4}{n}\binom{n}{2}=2n-2$$

G must be regular, $E(G)=2\lambda_1(G)$, and $E(\overline{G})=2\lambda_1(\overline{G})$ equality is only possible if all three conditions are met. If so, it must be $\lambda_2(G)\leq 0$ and $\lambda_2(\overline{G})\leq 0$ since the energy of a graph is equal to twice the sum of its positive eigenvalues. To put it another way, equality is only achieved if and only if both G and $\overline{G}$ have one and only one positive eigenvalue. Then, we must separately take into account each of the following three scenarios:

(i) G has a negative eigenvalue, meaning that $G\cong\overline{K}_n$.

(ii) G has no positive eigenvalue. i.e, $\overline{G}\cong\overline{K}_n$ indicating that $G\cong K_n$.

(iii) There is just one positive eigenvalue shared by both G and $\overline{G}$.

Smith's theorem [19] states that both G and $\overline{G}$ would then be full multipartite graphs. As opposed to their complements, which are disconnected, complete multipartite graphs are connected, making this impossible.

Corollary 3: Let B be the bipartite complement of $\overline{B}$, which is a bipartite graph with $a+b$ vertices. $E(B)+E(\overline{B})\geq 2\sqrt{ab}$ follows.

If and only if $B\cong K_{ab}$ or $B\cong\overline{K}_{a+b}$,equality is maintained.

Proof. The inequality is shown by noting that $A(B)+A(\overline{B})=A(K_{a.b})$ and $E(K_{a.b})=2\sqrt{ab}$. The case of equality is handled in a manner that is comparable to the proof of Corollary 2.

Corollary 4a: Assume that $G-e$ is the subgraph that results from eliminating the edge e from graph G. Then

$$E(G) \le E(G-e)+2 \qquad (5)$$

The outcome mentioned in this corollary 4a was already disclosed in [4]. The equality in (5) was demonstrated in [4] to hold only when e is an isolated edge in G. We obtain by repeatedly applying (5) to all edges of a (n,m)-graph

$$E(G) \le 2m \qquad (6)$$

with equality, which is also a previously established upper bound [1], if and only if G has m isolated edges and $n-2m$ isolated vertices. We obtain the following by repeatedly applying equation (5) to all edges of a (n,m)-graph, excluding those that end at a maximum-degree vertex:

Corollary 4b: If Δ is the highest degree at a vertex in a (n,m)- graph G, then

$$E(G) \le 2m - 2\left(\Delta - \sqrt{\Delta}\right) \qquad (7)$$

If and only if G is a union of the star $s_{\Delta+1}, m-\Delta$, m isolated edges, and $n-2m+\Delta+1$ isolated vertices, then equality in (7) holds.

Evidently, the bound (7) is superior to (6). It appears to be the first time it has been mentioned here. But if $m \ge 2n$, (7) is weaker than some other established upper bounds [9], such as McClleland's $\sqrt{2mn}$.

Similar to this, we also obtain the following energy upper bounds:

Corollary 4c. If G is a connected (n,m)-graph and T is its spanning tree, then $E(G) \le 2(m-n+1)+E(T)$.If G T, the inequality is rigid.

Corollary 4d. If G is a Hamiltonian (n,m)-graph, then $E(G) \le 2(m-n+1)+E(C_n)$ where C_n stands for the n-vertex cycle. If $G \cong C_n$, then the inequality is strict.

Corollary 4e. $E(G) \le 2(m-d)+E(P_{d+1})$,where P_k is the k-vertex path, if d is the diameter of a connected graph G. The inequality is strict if $G \cong P_{d+1}$.

We mention this in passing.

$$E(C_n) = \begin{cases} \dfrac{4\cos \pi/n}{\sin \pi/n} & if \;\; n = 0 \pmod 4 \\ \dfrac{4}{\sin \pi/n} & if \;\; n = 2 \pmod 4 \\ \dfrac{2}{\sin \pi/2n} & if \;\; n = 1 \pmod 2 \end{cases}$$

And

$$E(P_n)=\begin{cases}\dfrac{2}{\sin\dfrac{\pi}{2(n+1)}}-2 & if\ \ n=0(\bmod 2)\\[2ex] \dfrac{2\cos\dfrac{\pi}{2(n+1)}}{\sin\dfrac{\pi}{2(n+1)}}-2 & if\ \ n=1(\bmod 2)\end{cases}$$

III. Relating Laplacian energy and energy

We start by pointing out yet another straightforward Ky-Fan theorem consequence. Let d represent the graph's average vertex degree (n,m)-graph of G. Of course, $d=\frac{2m}{n}$.

Corollary 5. For a (n,m)-graph of G, where the vertex degrees are $d_1,d_2,\ldots,d_n$,and the average vertex degree is $\bar{d}$.

$$LE(G)\le E(G)=\sum_{i=1}^{n}\left|d_i-\bar{d}\right|$$

Proof. Rewrite Eq.(1) as

$$\left(L-\frac{2m}{n}I_n\right)=(-A)+\left(D-\frac{2m}{n}I_n\right)$$

Consider (4) as well as the fact that the diagonal matrix $D-\frac{2m}{n}I_n$ has eigenvalues of $d_i-d,i=1,2,\ldots,n$ when applying Theorem 1.

It was hypothesized in [10] that the Laplacian energy is always greater than or equal to the regular graph energy.

$$LE(G)\ge E(G) \tag{8}$$

Counter examples were ultimately employed to refute the validity of the conjecture [16,20]. Now, however, we demonstrate that this hypothesis was not entirely off the mark.

Theorem 6. If the graph G is bipartite, then relation (8) is true.

Proof. Eqs.(1) and (2) are subtracted, and the result is

$$L^{\dagger}-L=2A$$

Which can be written as

$$\left(L^{\dagger}-\frac{2m}{n}I_n\right)-\left(L-\frac{2m}{n}I_n\right)=2A \tag{9}$$

It is generally understood that the matrices L and $L^{\dagger}$ have equal spectra in the case of bipartite graphs (see, for example, [8]). In context with this, we briefly address

$$\sum_{i=1}^{n}s_i\left(L^{\dagger}-\frac{2m}{n}I_n\right)=\sum_{i=1}^{n}s_i\left(L-\frac{2m}{n}I_n\right)=\sum_{i=1}^{n}s_i\left(-\left[L-\frac{2m}{n}I_n\right]\right)=LE$$

and inequality (8) is obtained by using the Ky Fan theorem on equation (9).

We can arrive at a slightly more powerful result using similar reasoning. The result of adding (1) and (2) is in passing, we state

$$L^{\dagger}+L=2D$$

From which

$$\left(L^{\dagger}-\frac{2m}{n}I_n\right)+\left(L-\frac{2m}{n}I_n\right)=2\left(D-\frac{2m}{n}I_n\right)$$

Then by theorem1.

$$LE\geq\sum_{i=1}^{n}\left|d_i-\overline{d}\right|$$

The above inequality, along with the conclusions of Corollary 5 and Theorem 6, result in:

Theorem 7. For a bipartite (n,m)- graph G with vertex degrees $d_1,d_2,...,d_n$ and an average vertex degree of $d={}^{2m}\!/_{n}$

$$\max\left\{\sum_{i=1}^{n}\left|d_i-\overline{d}\right|\right\}\leq EL(G)\leq E(G)+\sum_{i=1}^{n}\left|d_i-\overline{d}\right|$$

IV. A difference in the intensity of the coalescence of two graphs.

Two graphs with disjoint vertex sets are G and H. Allow $u=V(G)$ and $v=V(H)$, by figuring out the vertices u and v, create the graph $G\circ H$ using copies of G and H. Consequently $\left|V(G\circ H)\right|=\left|V(G)\right|+\left|V(H)\right|-1$ The graph $G\circ H$ is referred to as the G and H coalescence with respect to u and v.

Theorem 8: Assume that G, H and $G\circ H$ are the afore mentioned graphs. Then

$$E(G\circ H)\leq E(G)+E(H) \tag{10}$$

If and only if either u is an isolated vertex of G or v is an isolated vertex of H, or both, equality is achieved.

Proof. When the vertices of the graphs G and H are appropriately labeled, the adjacency matrix of G and H takes the form

$$A=A(G\circ H)=\begin{bmatrix}R & x & 0\\ x^T & 0 & y^T\\ 0 & y & s\end{bmatrix}=\begin{bmatrix}R & x & 0\\ x^T & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & y^T\\ 0 & y & s\end{bmatrix}=B+C$$

where x is the column vector corresponding to the vertex, and $R=A(G-u)$ and $S=A(H-v)$.

y is the column vector corresponding to the vertex v in H, and u in G.

Then $E(G\circ H)=\sum_i s_i(A),E(G)=\sum_i s_i(B)$. Theorem 1 now follows directly from Relation (10).

Since either x or y is a zero vector in the equality case, it is simple to verify that the condition is sufficient.

The equality in (10) implies the equality in the singular value inequality for the necessity section. There is an orthogonal matrix P in the equality case of Theorem 1 such that both PB and PC are positive semi-definite. Let P now divide in the way described below using the matrix $A(G \circ H)$.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

where I stands for an identity matrix that is the proper size. Recall that both

$$P_{11}^T P_{11} + P_{21}^T P_{21} + P_{31}^T P_{31} = 1 \quad (11)$$

$$P_{13}^T P_{13} + P_{23}^T P_{23} + P_{33}^T P_{33} = 1 \quad (12)$$

$$P_{11}^T P_{13} + P_{21}^T P_{23} + P_{31}^T P_{33} = 0 \quad (13)$$

Where 1 denotes an identity matrix of appropriate size. Note that both

$$PB = \begin{bmatrix} P_{11}R + P_{12}x^T & P_{11}x & 0 \\ P_{21}R + P_{22}x^T & P_{21}x & 0 \\ P_{31}R + P_{32}x^T & P_{31}x & 0 \end{bmatrix}$$

And

$$PC = \begin{bmatrix} 0 & P_{13}y & P_{12}y^T + P_{13}S \\ 0 & P_{23}y & P_{22}y^T + P_{23}S \\ 0 & P_{33}y & P_{32}y^T + P_{33}S \end{bmatrix}$$

symmetry dictates that $P_{31}x = 0$ and $P_{13}y = 0$ because and are positive semi-definite. At this point, by multiplying equation (13) by x^T from the left and y from the right, we get

$$(P_{12}x)^T (P_{23}y) = x^T P_{11}^T P_{13} y + x^T P_{21}^T P_{23} y + x^T P_{31}^T P_{33} y = 0$$

Hence, one of the two scalars $(P_{12}x)^T$ and $(P_{23}y)$ must be zero.

Case 1. $(P_{12}x)^T = 0$

The positive semi-definite matrix PB is diagonal entry $P_{21}x$ should be noted. Due to the fact that $P_{21}x$ belongs in the entire column, which is zero, and so $P_{11}x = 0$.

Finally, $x^T x = x^T P_{11}^T P_{11} x + x^T P_{21}^T P_{21} x + x^T P_{31}^T P_{31} x = 0 + 0 + 0 = 0, i.e, x = 0$. This is due to Eq. (11), which states that $x = 0$. This indicates that G is isolated vertex of u.

Case 2. $(P_{23}y) = 0$

The positive semi-definite matrix PC is diagonal entry $P_{23}y$ should be noted. Due to the fact that $P_{21}x$ belongs in the entire column, which is zero, and so $P_{13}y = 0$. Finally,

$y^T y = y^T P_{13}^T P_{13} y + y^T P_{23}^T P_{23} y + y^T P_{33}^T P_{33} y = 0+0+0=0, i.e, y=0$. This is due to Eq. (12), which states that $y=0$. This indicates that H is isolated vertex of v.

The idea of hypo energetic graphs was just recently developed [11, 13]. $E(G)<n$ indicates that a graph G of order n is hypo energetic. For the purposes of the discussion at hand, we designate a graph as strongly hypo energetic if $E(G)<n-1$. (Remember that the n-vertex star for $n \geq 5$ is strongly hypo energetic.) Theorem 8 then yields the next:

Corolarlly 9a: Assume that the graphs G, H and $G \circ H$ are as in Theorem 8. If G and H are both strongly hypo energetic, then $G \circ H$ is also strongly hypo energetic.

Proof. From(10) and the fact that $E(G)<|V(G)|-1$ and $E(H)<|V(H)|-1$ follows:

$E(G \circ H)<|V(G)|+|V(H)|-2=|V(G \circ H)|-1$

Corolarlly 9b: If G and H are both strongly hypo energetic (or vice versa), then $G \circ H$ are both hypo energetic.

Conclusion

Graph energy and Laplacian graph energy, which are the sums of the absolute price, the set of special prices, the adjacency matrix, and the Laplacian matrix. Regular r-graphs and single-round linked graphs with no hanging vertices are equal, but in all other Laplacian graphs, the energy of the graph is larger than or equal to the energy of the graph.

The energy of the graph is always smaller than its two sides if the number of vertices in the graph $G(n,m)$ is higher than two sides, and this property does not occur in the Laplacian energy graph.

References

[1] G. Caporossi, D. Cvetkovi ´c, I. Gutman, P. Hansen, Variable neighborhood search for extremal graphs: 2. Finding graphs with extremal energy,J. Chem. Inf. Comput. Sci., 39 (1999) 984–996.

[2] D. Cvetkovi ´c, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, third ed., Johann Ambrosius Barth Verlag, Heidelberg, Leipzig, 1995.

[3] J. Day, W. So, Singular value inequality and graph energy change, Electron. J. Linear Algebra 16 (2007) 291–299.

[4] J. Day, W. So, Graph energy change due to edge deletion, Linear Algebra Appl. 428 (2008) 2070–2078.

[5] K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, Proc. Natl. Acad. Sci. USA 37 (1951) 760–766.

[6] I. Gohberg, M. Krein, Introduction to the Theory of Linear Non-selfadjoint Operators, Am. Math. Soc., Providence, 1969.

[7] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discrete Math. 7 (1994) 221–229.

[8] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.

[9] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.

[10] I. Gutman, N.M.M. de Abreu, C.T.M. Vinagre, A.S. Bonifácio, S. Radenkovi ´c, Relation between energy and Laplacian energy, MATCH Commun. Math. Comput. Chem. 59 (2008) 343–354.

[11] I. Gutman, X. Li, Y. Shi, J. Zhang, Hypoenergetic trees, MATCH Commun. Math. Comput. Chem. 60 (2008) 415–426.

[12] I. Gutman, O.E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Verlag, Berlin, 1986 (Chapter 8).

[13] I. Gutman, S. Radenkovi ´c, Hypoenergetic molecular graphs, Indian J. Chem. 46A (2007) 1733–1736.

[14] I. Gutman, B. Zhou, Laplacian energy of a graph, Linear Algebra Appl. 414 (2006) 29–37.

[15] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1989.

[16] J. Liu, B. Liu, On the relation between energy and Laplacian energy, MATCH Commun. Math. Comput. Chem. 61 (2009) 403–406.

[17] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi, G. Chartrand, O.R. Oellermann, A.J. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications, Wiley, New York, 1991, pp. 871–898.

[18] V. Nikiforov, The energy of graphs and matrices, J. Math. Anal. Appl. 326 (2007) 1472–1475.

[19] J.H. Smith, Some properties of the spectrum of a graph, in: R. Guy, H. Hanani, N. Sauer, J. Schönheim (Eds.), Combinatorial Structures and Their Applications, Gordon and Breach, New York, 1970, pp. 403–406.

[20] D. Stevanovi ´c, I. Stankovi ´c, M. Miloševi ´c, More on the relation between energy and Laplacian energy of graphs, MATCH Commun. Math. Comput. Chem. 61 (2009) 395–401.

[21] B. Zhou, I. Gutman, On Laplacian energy of graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 211–220.