# Solution of biological population by fractional generalized homotopy analysis method

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**Abstract:** This paper aims to solve the Biological Population model problem using a hybrid method called fractional generalized homotopy analysis method (FGHAM). The fractional derivatives are described by Caputo?s sense. The method introduces a significant improvement in this field over existing techniques. Results obtained using the scheme presented here agree well with the analytical solutions and the numerical results presented in [1]. The convergence region of the Biological Population model solutions are clearly identified using form series solutions are produced using FGHAM. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

**Keywords:** fractional biological population equation, homotopy generalized analysis methd, fractional claculus, mittag-leffer function

*Introduction*

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical, chemical processes, biology and engineering for example [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Recently, several mathematical methods have been developed to obtain exact and approximate analytic solutions [15], [1], [13], [14], [16], [17], [18], [17], [19]. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations and some other methods give the solution in a series form, which converges to the exact solution. An effective and easy to use method for solving such equations is needed, the Generalized homotopy analysis method (GHAM) is a successful method to find the exact analytical solutions for linear and nonlinear problem. GHAM successfully applied into physics, engineering fields, [20], [21], [22], [23], [24].

In this paper, we implement the homotopy analysis method (GHAM) to this model with some initial conditions to find explicit solutions and numerical solutions, rather than the traditional methods. The GHAM scheme is illustrated by studying the biological population model to compute explicit and numerical solutions.

The main aim of this paper is to solve the nonlinear fractional-order biological population model in the form

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = \frac{\partial^2 (v^2)}{\partial x^2} + \frac{\partial^2 (v^2)}{\partial y^2} + f(v). \quad (1)$$

with given initial condition $u(x, y, 0)$, where u denotes the population density and f represents the population supply due to births and deaths. We implement the homotopy analysis method (GHAM)to this model subject to the initial conditions

$$v(x.y.0) = f_0(x.y).$$

*Basic definitions*

This section discusses some basic definitions of fractional calculus used in this study. The Riemann-Liovillie fractional integrals of the left and right sides are defined for any function $\phi(x) \in L_1(a.b)$ as:

$$(I_{a+}^{\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^{\infty} (x-t)^{\alpha-1} \phi(t) dt. x > a.$$

$$(I_{a-}^{\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{b} (x-t)^{\alpha-1} \phi(t) dt. x < b.$$

The left- and right-handed Riemann-Liouville fractional derivatives of order $\alpha. 0 < \alpha < 1$, in the interval $[a.b]$ are defined as:

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt,$$

$$(D_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (x-t)^{-\alpha} f(t) dt,$$

The Caputo fractional derivative of order $\alpha$ is defined as:

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi$$

where $m - 1 < \alpha \leq m. m \in \mathbb{N}$.

The Mittag-Leffler function, which is a generalization of the exponential function, is defined as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

where $\alpha \in \mathbb{C}. R(\alpha) > 0$.

The continuous function $f: \mathbb{R} \to \mathbb{R}. t \to f(t)$ has a fractional derivative of order $k\alpha$. For any positive integer k and for any $\alpha$, $0 < \alpha < 1$, the Taylor series of fractional order is given by:

$$f(t+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(t). 0 < \alpha < 1$$

where $\Gamma(1 + \alpha k) = (\alpha k)!$

Let $f(t)$ be a continuous function. Then, the solution $y(t). y(0) = 0$ is given by

$$y = \int_0^t f(\xi)(d\xi)^{\alpha} = \alpha \int_0^t (t-\xi)^{(\alpha-1)} f(\xi) d\xi. 0 < \alpha < 1$$

*Fractional generalized integral transform (fractional G-transform)*

In this section introduces the fractional G-transform and show some of the properties which proposed by [?]

Suppose $g(t)$ be any time-domain function defined for $t > 0$. Then, the fractional G-transform of order $\alpha$ of $g(t)$ is denoted by $G_\alpha[g(t)]$ and is defined as:

$$\begin{aligned} G_\alpha[g(t)] = H_\alpha[u] \quad &= u^{p+1}\int_0^\infty g(ut)E_\alpha(-t)^\alpha(dt)^\alpha \\ &= u^{p-\alpha+1}\int_0^\infty g(t)E_\alpha\left(\frac{-t}{u}\right)^\alpha(dt)^\alpha \\ &= \lim_{M\to\infty} u^{p-\alpha+1}\int_0^M g(t)E_\alpha\left(\frac{-t}{u}\right)^\alpha(dt)^\alpha \end{aligned}$$

which $E_\alpha$ is the Mittag-Leffler function. The fractional G-transform satisfies the following properties: If the Laplace Transform of fractional order of a function $g(t)$ is $\mathcal{L}_\alpha g(t) = F_\alpha(s)$, the fractional G-transform of order $\alpha$ of $g(t)$ is

$$G_\alpha[g(t)] = H_\alpha[u] = u^{p-\alpha+1}F_\alpha\left(\frac{1}{u}\right).$$

If $G_\alpha[g(t)] = H_\alpha[u]$. then

$$G_\alpha[g(t)] = \frac{1}{a^\alpha}H_\alpha\left[\frac{u}{a}\right].$$

where $a$ is a non-zero constant.

If $G_\alpha[g(t)] = H_\alpha[u]$. then

$$G_\alpha[g(t-b)] = E_\alpha\left(\frac{-b}{u}\right)^\alpha H_\alpha(u).$$

If $G_\alpha[g(t)] = H_\alpha[u]$. then

$$G_\alpha[E_\alpha(a^\alpha t^\alpha)g(t)] = \left(\frac{1}{1-au^2}\right)^\alpha H_\alpha\left(\frac{u}{1-au}\right).$$

The systematic procedure for the FGHAM [?] is given in the next section.

*Fractional generalized homotopy analysis method (FGHAM)*

Consider a fractional non-linear partial differential equation with the following initial condition:

$$D^\alpha v(x.y.t) + Rv(x.y.t) + Nv(x.y.t) = g(x.y.t), v(x.y.0) = f(x.y). \qquad (2)$$

where Dα is the fractional differential operator $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, R is the linear differential operator, N is the non-linear differential operator, and $g(x.t)$ is the source term. The following systematic procedure steps are used to solve the non-linear fractional differential equations:

1. Using fractional G-transform, (2) is transformed to

$$G_\alpha[D^\alpha v(x.y.t)] + G_\alpha[Rv(x.y.t)] + G_\alpha[Nv(x.y.t)] = G_\alpha[g(x.y.t)]. \qquad (3)$$

2. Applying the derivative property of fractional G-transform, (3) is expressed as:

$$\begin{aligned} &G_\alpha[v(x.y.t)] - u^{p+1}v_0(x.y.t) \\ &+u^\alpha(G_\alpha[Rv(x.y.t)] + G_\alpha[Nv(x.y.t)] - G_\alpha[g(x.y.t)]) = 0. \end{aligned} \qquad (4)$$

3. Decomposing the non-linear terms in (4), the following homotopy is constructed:

$$(1-s)G_\alpha[\phi(x.y.t;s)-v_0(x.y.t)] = hsH(x.y.t)N[\phi(x.y.t;s)]. \quad (5)$$

where $s \in [0.1]$ is an embedding parameter and $\phi(x.y.t;s)$ is a real function of $x.y.t.$ and $s$, while $h$ is a non-zero auxiliary parameter, $H(x.y.t) \neq 0$ is an auxiliary function, $v_0(x.y.t)$ is an initial condition of v(x,y,t), and $\phi(x.y.t;s)$ is a unknown function. (5) is called the zero-order deformation equation.

In (5), if $s = 0$ and $s = 1$, then $\phi(x.y.t;0) = v_0(x.y.t)$ and $\phi(x.y.t;1) = v(x.y.t)$, respectively.

If $s \in [0.1]$, then the solution is transferred from $v_0(x.y.t)$ to $v(x.y.t)$.

4. Deriving the $n^{th}$-order deformation equation is following as:

$$G_\alpha[v_n(x.y.t)-\chi_n v_{n-1}(x.y.t)] = hH(x.y.t)R_n(\mathbf{v_{n-1}}(x.y.t)). \quad (6)$$

5. Using the Inverse G-transform on both the sides of (6), the following equation is obtained:

$$v_n(x.y.t) = \chi_n v_{n-1}(x.y.t) + hG_\alpha^{-1}[H(x.y.t)R_n(\mathbf{v_{n-1}}.x.y.t)] \quad (7)$$

where

$$\begin{aligned} R_n(\mathbf{v_{n-1}}.x.y.t) = \quad & G_\alpha[v(x.y.t)] - u^{p+1}(1-\chi_n)v_0(x.y.t) \\ & +u^\alpha(G_\alpha[Rv(x.y.t)] + G_\alpha[Nv(x.y.t)] - G_\alpha[g(x.y.t)]). \end{aligned} \quad (8)$$

and

$$\chi = \begin{cases} 0.\, n \leq 1 \\ 1.\, n > 1. \end{cases}$$

6. The following solution is obtained:

$$v(x.y.t) = v_0(x.y.t) + \sum_{n=1}^{\infty} v_n(x.y.t). \quad (9)$$

*Fractional Biological population equation*

In this section, we present some examples with analytical solution to show the efficiency of methods described in the previous section.

Example 5.1: Considering the fractional Biological population equation:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v(1-rv).$$

with initial condition

$$v(x.y.0) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right].$$

Applying the fractional G-transform on both the side of equation (5.1)

$$G_\alpha\left[\frac{\partial^\alpha v}{\partial t^\alpha}\right] = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v(1-rv)\right]$$

$$\frac{1}{u^\alpha}G_\alpha[v(x.y.t)] - \frac{1}{u^{\alpha-1}}v(x.y.0)u^p = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v(1-rv)\right]$$

$$G_\alpha[v(x.y.t)] - u^{p+1}v(x.y.0) - u^\alpha G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v(1-rv)\right] = 0.$$

Applying FGHAM:

$$v_n(x.y.t) = \chi_n v_{n-1}(x.y.t) + hG_\alpha^{-1}[R_n(\mathbf{v_{n-1}}.x.y.t)].$$

where

$$R_n(\mathbf{v_{n-1}}.x.y.t) = G_\alpha[v_{n-1}(x.y.t)] - (1-\chi_n)u^{p+1}v(x.y.0)$$
$$-u^\alpha G_\alpha\left[\frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial x^2} + \frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial y^2} + v_{n-1}(x.y.t)\left(1 - rv_{n-1}(x.y.t)\right)\right].$$

Solving the above equation for $n = 1.2.3. \ldots$:

$$v(x.y.0) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right].$$
$$v_1(x.y.t) = -h\frac{t^\alpha}{\Gamma(\alpha+1)}\left[\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\right]$$
$$v_2(x.y.t) = -h(h+1)\frac{t^\alpha}{\Gamma(\alpha+1)}\left[\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\right] + h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\right]$$
$$v_3(x.y.t) = (1+h)v_2(x,y,t) + h^2(h+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\right]$$
$$-h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left[\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\right]$$

Similarly, $v_4.v_5. \ldots.$ are estimated and the series solution is obtained, that is:

$$v(x.y.t) = v_0(x.y.t) + \sum_{n=1}^{\infty} v_n(x.y.t). \quad (10)$$

If $h = -1$ (10) can be expressed as:

$$v(x.y.t) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]E_\alpha(t^\alpha) \quad (11)$$

If we put $\alpha = 1$, we obtained the exact solution

$$v(x.y.t) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]e^t = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)+t\right]. \quad (12)$$

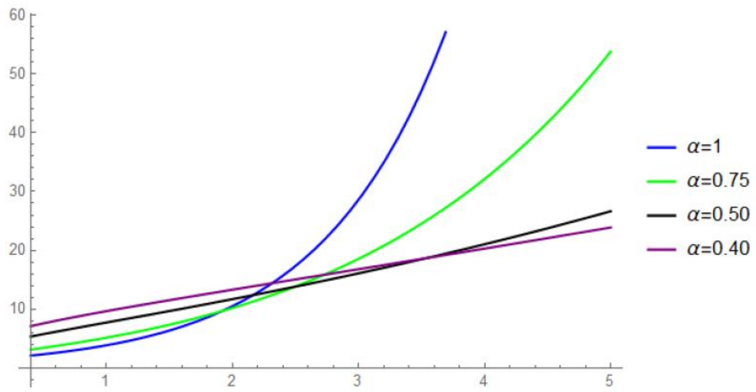


Figure 1: A graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha = 1.0.75.0.50.0.40$.

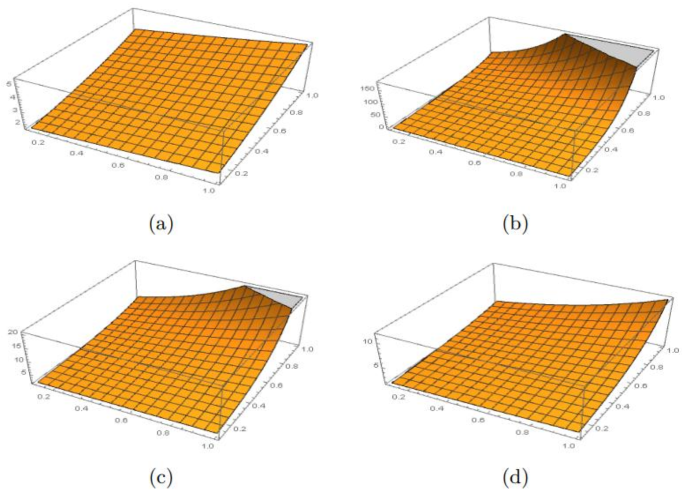


Figure 2: The solution for Biological Population equation subject to various setting of fractional parameter $\alpha = 10.75.0.50.0.40$ and auxiliary parameter $h = -1$ are shown in a-d,respectivel

| t | $Ev_3$ | $Ev_4$ | $Ev_5$ |
|---|---|---|---|
| 0.1 | 0.0002500 | 0.00001600 | 9.1000000E − 7 |
| 0.2 | 0.00210000 | 0.00023000 | 0.00002100 |
| 0.3 | 0.00750000 | 0.00110000 | 0.00014000 |
| 0.4 | 0.01800000 | 0.00330000 | 0.00051000 |
| 0.5 | 0.03700000 | 0.00780000 | 0.00140000 |
| 0.6 | 0.06700000 | 0.01600000 | 0.00340000 |
| 0.7 | 0.11000000 | 0.02900000 | 0.00690000 |
| 0.8 | 0.17000000 | 0.05000000 | 0.01300000 |
| 0.9 | 0.25000000 | 0.08100000 | 0.02300000 |
| 1 | 0.36000000 | 0.12000000 | 0.03700000 |

Table 1: Absolute error for the Biological Population equation

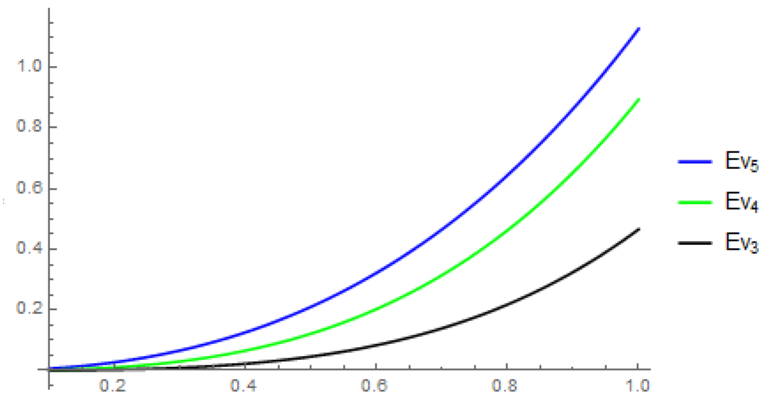


Figure 3: An error graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha =$ 1.0.75.0.5.0.40.

Example 5.2: Considering the following generalized Fractional Biological population equation:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 (v^2)}{\partial x^2} + \frac{\partial^2 (v^2)}{\partial y^2} + kv. \qquad (13)$$

with the initial condition

$$v(x.y.0) = \sqrt{xy}.$$

Applying the fractional G-transform on both the sides of (13)

$$G_\alpha\left[\frac{\partial^\alpha v}{\partial t^\alpha}\right] = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + kv\right]$$

$$\frac{1}{u^\alpha}G_\alpha[v(x.y.t)] - \frac{1}{u^{\alpha-1}}v(x.y.0)u^p = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + kv\right]$$

$$G_\alpha[v(x.y.t)] - u^{p+1}v(x.y.0) - u^\alpha G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + kv\right] = 0.$$

Applying FGHAM:

$$v_n(x.y.t) = \chi_n v_{n-1}(x.y.t) + hG_\alpha^{-1}[R_n(\mathbf{v_{n-1}}.x.y.t)].$$

where

$$R_n(\mathbf{v_{n-1}}.x.y.t) = G_\alpha[v_{n-1}(x.y.t)] - (1-\chi_n)u^{p+1}v(x.y.0)$$
$$-u^\alpha G_\alpha\left[\frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial x^2} + \frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial y^2} + kv_{n-1}(x.y.t)\right].$$

Solving the above equation for $n = 1.2.3. ...$:

$$v(x.y.0) = \sqrt{xy}.$$
$$v_1(x.y.t) = -kh\frac{t^\alpha}{\Gamma(\alpha+1)}\sqrt{xy}$$
$$v_2(x.y.t) = -kh(h+1)\frac{t^\alpha}{\Gamma(\alpha+1)}\sqrt{xy} + k^2h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{xy}$$
$$v_3(x.y.t) = k(1+h)v_2(x.y.t) + k^2h^2(h+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{xy} - k^3h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sqrt{xy}$$

Similarly, $v_4.v_5. ....$ are estimated and the series solution is obtained, that is:

$$v(x.y.t) = v_0(x.y.t) + \sum_{n=1}^{\infty} v_n(x.y.t). \tag{14}$$

If $h = -1$ (14) can be expressed as:

$$v(x.y.t) = \sqrt{xy}\sum_{n=0}^{\infty}\frac{k^n t^{n\alpha}}{\Gamma(n\alpha+1)} = \sqrt{xy}E_\alpha(kt^\alpha) \tag{15}$$

If we put $\alpha = 1$, we obtained the exact solution

$$v(x.y.t) = \sqrt{xy}e^{kt} \tag{16}$$

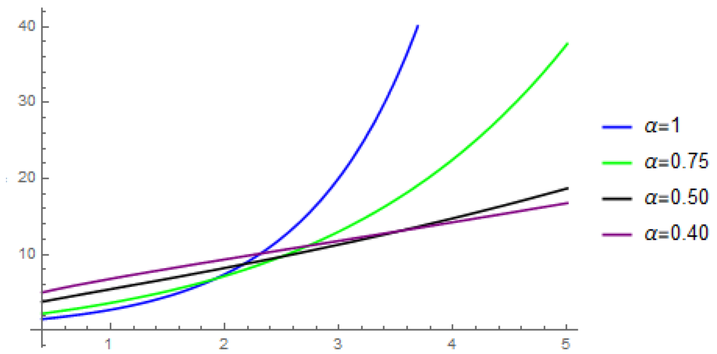


Figure 4: A graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha = 1, 0.75, 0.50, 0.40$.

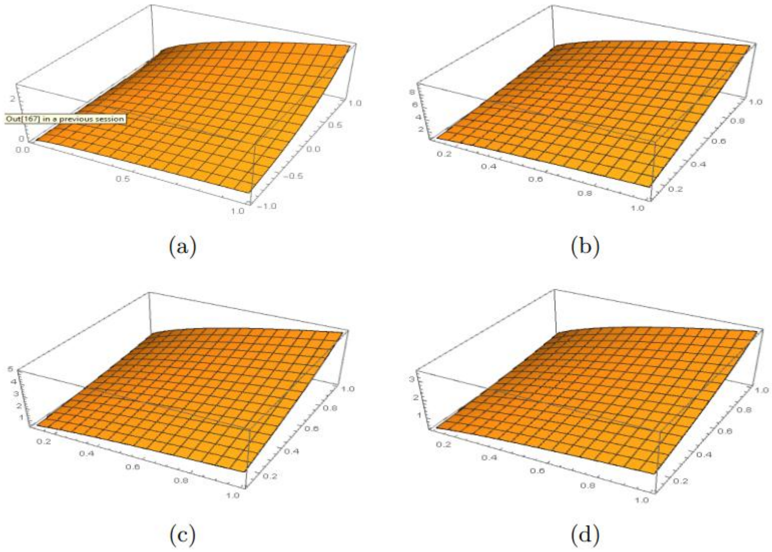


Figure 5: The solution for Biological Population equation subject to various setting of fractional parameter $\alpha = 1, 0.75, 0.50, 0.40$ and auxiliary parameter $h = -1$ are shown in a-d,respectively

| t | $Ev_3$ | $Ev_4$ | $Ev_5$ |
|---|---|---|---|
| 0.1 | 0.00620000 | 0.00160000 | 0.00028000 |
| 0.2 | 0.02700000 | 0.01000000 | 0.00250000 |
| 0.3 | 0.06600000 | 0.03000000 | 0.00890000 |
| 0.4 | 0.13000000 | 0.06600000 | 0.02200000 |
| 0.5 | 0.21000000 | 0.12000000 | 0.04600000 |
| 0.6 | 0.32000000 | 0.20000000 | 0.08400000 |
| 0.7 | 0.47000000 | 0.31000000 | 0.14000000 |
| 0.8 | 0.65000000 | 0.46000000 | 0.22000000 |
| 0.9 | 0.86000000 | 0.46000000 | 0.32000000 |
| 1 | 1.10000000 | 0.89000000 | 0.47000000 |

Table 2: Absolute error for the Biological Population equation

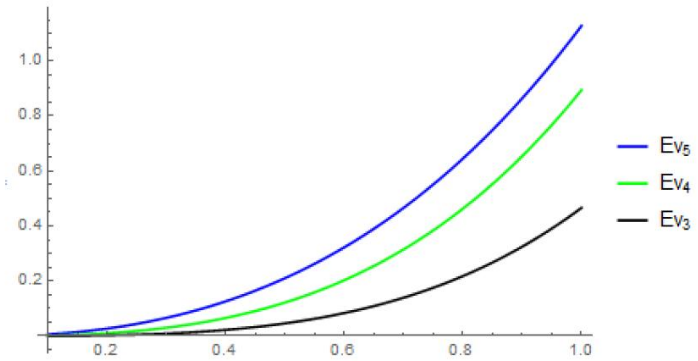


Figure 6: An error graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha = 1, 0.75, 0.50, 0.40$.

Considering the following generalized Fractional Biological population equation:

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = \frac{\partial^2 (v^2)}{\partial x^2} + \frac{\partial^2 (v^2)}{\partial y^2} + v. \quad (17)$$

with the initial condition

$$v(x, y, 0) = \sqrt{\mathrm{sinxsinhy}},$$

Applying the fractional G-transform on both the sides of (20)

$$G_\alpha\left[\frac{\partial^\alpha v}{\partial t^\alpha}\right] = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v\right]$$

$$\frac{1}{u^\alpha}G_\alpha[v(x.y.t)] - \frac{1}{u^{\alpha-1}}v(x.y.0)u^p = G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v\right]$$

$$G_\alpha[v(x.y.t)] - u^{p+1}v(x.y.0) - u^\alpha G_\alpha\left[\frac{\partial^2(v^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + v\right] = 0.$$

Applying FGHAM:

$$v_n(x.y.t) = \chi_n v_{n-1}(x.y.t) + hG_\alpha^{-1}[R_n(\mathbf{v_{n-1}}.x.y.t)].$$

where

$$R_n(\mathbf{v_{n-1}}.x.y.t) = G_\alpha[v_{n-1}(x.y.t)] - (1-\chi_n)u^{p+1}v(x.y.0)$$
$$-u^\alpha G_\alpha\left[\frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial x^2} + \frac{\partial^2\left(v_{n-1}^2(x.y.t)\right)}{\partial y^2} + v_{n-1}(x.y.t)\right].$$

Solving the above equation for $n = 1,2,3,\ldots$:

$$v(x.y.0) = \sqrt{\mathrm{sinxsinhy}}.$$
$$v_1(x.y.t) = -h\frac{t^\alpha}{\Gamma(\alpha+1)}\sqrt{\mathrm{sinxsinhy}}$$
$$v_2(x.y.t) = -h(h+1)\frac{t^\alpha}{\Gamma(\alpha+1)}\sqrt{\mathrm{sinxsinhy}} + h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{\mathrm{sinxsinhy}}$$
$$v_3(x.y.t) = (1+h)v_2(x.y.t) + h^2(h+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{\mathrm{sinxsinhy}} - h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sqrt{\mathrm{sinxsinhy}}$$

Similarly, $v_4.v_5.\ldots.$ are estimated and the series solution is obtained, that is:

$$v(x.y.t) = v_0(x.y.t) + \sum_{n=1}^{\infty} v_n(x.y.t), \tag{18}$$

If $h = -1$ (18) can be expressed as:

$$v(x, y, t) = \sqrt{\mathrm{sinxsinhy}}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \sqrt{\mathrm{sinxsinhy}}E_\alpha(t^\alpha). \tag{19}$$

If we put $\alpha = 1$, we obtained the exact solution

$$v(x.y.t) = \sqrt{\mathrm{sinxsinhy}}e^t. \tag{20}$$

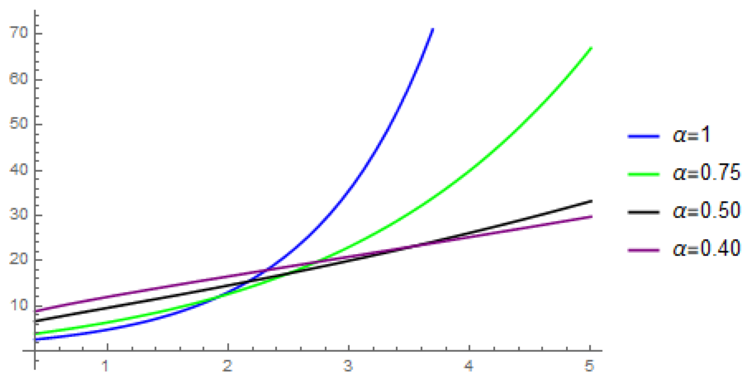


Figure 7: A graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha = 1, 0.75, 0.50, 0.40$.

| t | $Ev_3$ | $Ev_4$ | $Ev_5$ |
|---|---|---|---|
| 0.1 | 0.04300000 | 0.01200000 | 0.00320000 |
| 0.2 | 0.15000000 | 0.05400000 | 0.01900000 |
| 0.3 | 0.31000000 | 0.13000000 | 0.05400000 |
| 0.4 | 0.54000000 | 0.26000000 | 0.12000000 |
| 0.5 | 0.84000000 | 0.43000000 | 0.21000000 |
| 0.6 | 1.20000000 | 0.67000000 | 0.35000000 |
| 0.7 | 1.70000000 | 0.97000000 | 0.53000000 |
| 0.8 | 2.20000000 | 1.30000000 | 0.78000000 |
| 0.9 | 2.90000000 | 1.80000000 | 1.10000000 |
| 1 | 3.60000000 | 2.40000000 | 1.50000000 |

Table 3: Absolute error for the Biological Population equation

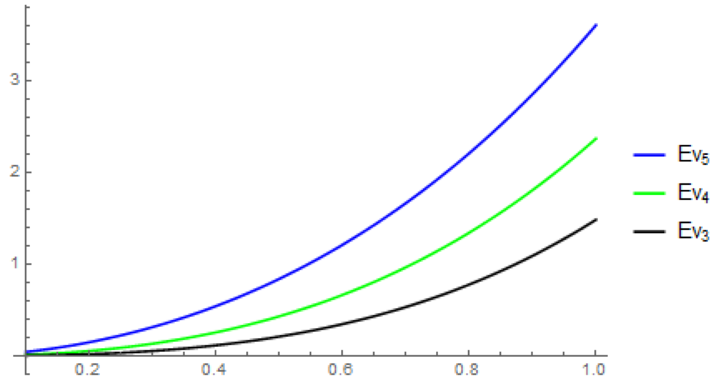


Figure 9: An error graphical illustration of the Solution of Biological Population equation using various setting of integer and fractional parameter $\alpha = 1,0.75,0.5,0.40$.

*Conclusion*

Employ the FGHAM for finding the exact solutions of generalized biological populations equation subject to some initial conditions. Results obtained using the scheme presented here agree well with the analytical solutions and the numerical results presented in [1], [25] by Adomian?s decomposition method is ADM and RVIM. However, in [26] it was shown that ADM does not converge in general, in particular, when the method is applied to linear operator equations. It was also shown that ADM is equivalent to Picard iteration method, and therefore it might diverge. The Homotopy Analysis Method is another technique used to derive an analytic solution for nonlinear operators. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter h, auxiliary function H(t), and auxiliary linear operator L. It is apparently seen that FGHAM is a very powerful and efficient technique in finding analytical solutions for wide classes of differential equations. They also do not require large computer memory.

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