Sonning butun va kasr qismi qatnashgan tenglamalarni yechish yuzasidan ba'zi metodik tavsiyalar

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Annotatsiya: Maqolada sonning butun va kasr qismi $y=[x]$, $y=\{x\}$ funksiyalari va ularning grafiklari, xossalari haqida ma'lumotlar bayon etilgan. Sonning butun va kasr qismi qatnashga tenglamalarni yechish yuzasidan bir qancha metodik tavsiyalar keltirib o'tilgan.

Kalit so'zlar: sonning butun va kasr qismi, ta'rif, funksiya, xossa, isbot, aniq integral, masala va yechim

Some methodological recommendations for solving equations involving whole and fractional numbers

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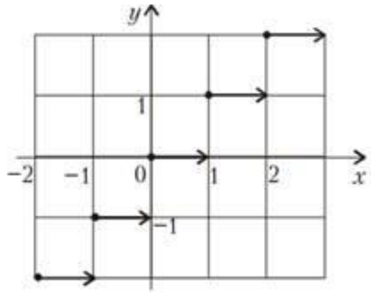
Abstract: The article describes information about the functions $y=[x]$, $y=\{x\}$ and their graphs and properties. A number of methodical recommendations for solving equations involving whole and fractional parts of numbers are given.

Keywords: integer and fractional part of a number, definition, function, property, proof, definite integral, problem and solution

Ta'rif. Haqiqiy x sonning $[x]$ *butun qismi* deb, x dan katta bo'lmagan eng katta butun songa aytiladi.

Masalan: [-1,5]=-2; [-1]= -1; [0]=0; [1,5]=1; [π]=3.

Umuman olganda, ta'rifga ko'ra, $[x]=k$ tenglik quyidagini bildiradi: k son $k \le x < k+1$ shartni qanoatlantiradigan butun sondir.



1-rasm. y=[x] funksiyaning grafigi

$y=[x]$ funksiyaning grafigi zinasimon ko‘rinishga ega bo‘ladi (1-rasm).

$\{x\}=x-[x]$ tenglik bilan $x(x\in R)$ sonining *kasr qismi* aniqlanadi.

Masalan: $\{-0,2\}=0,8;\quad \left\{-\frac{1}{3}\right\}=\frac{2}{3};\quad \{\sqrt{2}\}=\sqrt{2}-1;\quad \{1\}=0$

Xossalari: Ixtiyoriy x va y haqiqiy sonlar uchun quyidagi tasdiqlar o‘rinli bo‘ladi.

1) $[x]\le x$

2) $[x+n]=[x]+n,\quad n\in Z$

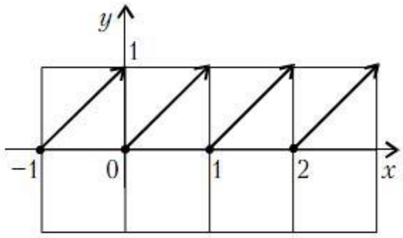
3) $[x+y]\ge[x]+[y]$

4) $\{x\}=x$ tenglik $0\le x<1$ bo‘lgan holdagina bajariladi

5) $\{x\}=\{y\}$ tenglik $x-y=n$ (bu yerda n-butun son) bo‘lgan holatda bajariladi

6) Ixtiyoriy x haqiqiy soni uchun $\{x+1\}=\{x\}$ bo‘ladi.

$y=\{x\}$ funksiya eng kichik davri 1 ga teng bo‘lgan davriy funksiyadir. Uning grafigi 2-rasmda keltirib o‘tilgan.



2-rasm. y={x} funksiyaning grafigi

Quyida sonning butun va kasr qismi qatnashgan ba’zi masalalarning yechimlarini keltirib o‘tamiz.

1-masala (II Soros olimpiadasi). $x^2-10[x]+9=0$ tenglamani yeching.

Yechim. Berilgan tenglamaning ildizi x bo‘lsin. $n=[x]$ deb olsak, u holda $x^2+9=10n$ bo‘ladi. Demak, $n>0$. Endi $n=[x]$ belgilashdan $n\le x<n+1$ ni yoza olamiz. Oxirgi musbat hadli tengsizlikni kvadratga ko‘tarib 9 ni hadma-had qo‘shsak, $n^2+9\le x^2+9<(n+1)^2+9=n^2+2n+10$ ni hosil qilamiz, $x^2+9=10n$ ekanligidan,

$n^2+9\le 10n<n^2+2n+10$ tengsizlikni hosil qilamiz. Endi bu tengsizlikka teng kuchli bo‘lgan quyidagi tengsizliklar sistemasini yozamiz va yechimni topamiz:

$$\begin{cases} n^2+9\le 10n \\ 10n<n^2+2n+10 \end{cases} \Rightarrow \begin{cases} 1\le n\le 9 \\ n<4-\sqrt{6}\ \text{ va }\ n>4+\sqrt{6} \end{cases} \Leftrightarrow 1\le n<4-\sqrt{6}\ \text{ yoki }\ 4+\sqrt{6}<n\le 9$$

Oxirgi natijadan va n - butun son ekanligidan n=1, 7, 8, 9 holatlar bo‘lishi mumkin. Bu qiymatlarni navbatma-navbat $x^2+9=10n$ ifodaga qo‘yib x ning mumkin bo‘lgan barcha qiymatlarini topamiz (bunda $n\le x<n+1$ ekanligidan $x>0$). Javob: $1;\ \sqrt{61};\ \sqrt{71};\ 7$. (N.Raximov, Maktab o‘quvchilarida sonning butun va kasr qismiga oid masalalarni yechich ko‘nikmasini shakllantirish. , 2022/3/26.)

2-masala. $\left\lfloor \frac{2x+1}{3} \right\rfloor = [x]$ tenglamani yeching.

Yechim. Faraz qilaylik, $[x]=k$ bo‘lsin. U holda,

$$\begin{cases} k \le \frac{2x+1}{3} < k+1 \\ k \le x < k+1 \end{cases} \Rightarrow \begin{cases} \frac{3k-1}{2} \le x < \frac{3k+2}{2} \\ k \le x < k+1 \end{cases}$$

Bundan k quyidagi tengsizliklarni qanoatlantirishi kelib chiqadi:

$$\frac{3k-1}{2} < k, \quad k < \frac{3k+2}{2}$$. Ya’ni: $-2<k<3$.

Shunday qilib, k ning qiymatlari -1; 0; 1; 2 bo‘lishi mumkin. Ushbu qiymatlarni ketma-ket yuqoridagi sistemaga qo‘yib, quyidagi javobni topamiz.

Javob: $-1 \le x < -\frac{1}{2}; \quad 0 \le x < 2; \quad \frac{5}{2} \le x < 3$. (N.Raximov, Maktab o‘quvchilarida sonning butun va kasr qismiga oid masalalarni yechich ko‘nikmasini shakllantirish. , 2022/3/26.)

3-masala. $[x^2] = 2[x]$ tenglamani yeching.

Yechim. Faraz qilaylik, $[x]=k, \ \{x\}=\alpha$ bo‘lsin. U holda, $k \ge 0, \ \alpha \ge 0$ va

$$[(k+\alpha)^2] = 2[k+\alpha].$$

Bundan esa quyidagi tenglamani hosil qilamiz: $[2k\alpha + \alpha^2] = 2k - k^2$. $k \ge 0, \ \alpha \ge 0$ bo‘lgani uchun bu tenglamaning chap tomoni manfiy emas.

Demak, $2k - k^2 \ge 0$ va k soni butun son bo‘lgani uchun u faqat 0, 1 yoki 2 qiymatlarga ega bo‘lishi mumkin.

1) $k=0$ bo‘lganda $0 \le \alpha < 1$. Bundan $[\alpha^2] = 0$ ni hosil qilamiz. Demak, $0 \le x < 1$ kelib chiqadi.

2) $k=1$ bo‘lganda quyidagi tenglamani hosil qilamiz: $[2\alpha + \alpha^2] = 1$. Bu esa $1 \le 2\alpha + \alpha^2 < 2, \ \ 0 \le \alpha < 1$ sistemani beradi, bundan $\sqrt{2} - 1 \le \alpha < 1, \ \ \sqrt{2} \le x < 2$ kelib chiqadi.

3) Nihoyat, $k=2$ bo‘lganda, $[4\alpha + \alpha^2] = 0$ tenglamaga ega bo‘lamiz, bu esa $0 \le 4\alpha + \alpha^2 < 1, \ \ 0 \le \alpha < 1$ sistemaga teng kuchlidir. Uning yechimi - $0 \le \alpha < \sqrt{5} - 2, \ \ 2 \le x < \sqrt{5}$ kelib chiqadi.

Hosil bo‘lgan $0 \le x < 1$, $\sqrt{2} \le x < 2$ va $2 \le x < \sqrt{5}$ yechimlarni birlashtirib umumiy javobni yozamiz. Javob: $0 \le x < 1$, $\sqrt{2} \le x < \sqrt{5}$.

4-masala (V Soros olimpiadasi). Tenglamalar sistemasini yeching.

$$\begin{cases} x+[y]+\{z\}=3{,}9 \\ y+[z]+\{x\}=3{,}5 \\ z+[x]+\{y\}=2 \end{cases}$$

Yechim. Faraz qilaylik,

$a=[x],\ \alpha=\{x\},\ b=[y],\ \beta=\{y\},\ c=[z],\ \gamma=\{z\}$ bo‘lsin. bu yerda *a,b,c* - butun sonlar, $0\le\alpha<1,\ 0\le\beta<1,\ 0\le\gamma<1.$ Ushbu belgilashlardan so‘ng berilgan sistema quyidagi shaklda ifoda qilinadi: $\begin{cases} a+\alpha+b+\gamma=3{,}9 \\ b+\beta+c+\alpha=3{,}5 \\ c+\gamma+a+\beta=2 \end{cases}$

Bu tenglamalarni qo‘shib quyidagi tenglikni hosil qilamiz:

$$2(a+b+c+\alpha+\beta+\gamma)=9{,}4$$

$$a+b+c+\alpha+\beta+\gamma=4{,}7$$

Hosil bo‘lgan oxirgi tenglamadan sistemadagi birinchi, ikkinchi va uchinchi tenglamalarni ketma-ket ayirib quyidagilarga ega bo‘lamiz: $\begin{cases} c+\beta=0{,}8 \\ a+\gamma=1{,}2 \\ b+\alpha=2{,}7 \end{cases}$

bundan $c=0;\ \beta=0{,}8;\ a=1;\ \gamma=0{,}2;\ b=2;\ \alpha=0{,}7$ ekanligi kelib chiqadi.

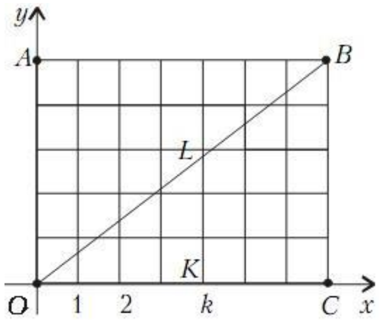
Javob: *x*=1,7; *y*=2,8; *z*=0,2.

5-masala. *p* va *q* - o‘zaro tub butun sonlar uchun

$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+\ldots+\left[\frac{(q-1)p}{q}\right]=\frac{(p-1)(q-1)}{2}$$

formula o‘rinli ekamnligini isbotlang.

Yechim. *XOY* tekislikda butun koordinatali (*x;y*) nuqtalar to‘plamini qaraymiz, bunda $1\le x\le q-1,\quad 1\le y\le p-1$ shart bajarilsin.



3-rasm

Bu to‘plam *OABC* to‘g‘ri to‘rtburchakning ichidan olingan bo‘lib (3-rasm), jami $(p-1)(q-1)$ ta butun koordinatali nuqtalarga ega. Ushbu to‘g‘ri to‘rtburchakning diagonalida *O* va *B* nuqtalardan boshqa butun koordinatalarga ega bo‘lgan nuqtalar mavjud emas. Haqiqatdan ham, agar butun koordinatali (*t; p*) nuqta *OB* da yotsa (bu yerda 1<m<q), u holda $tg\angle BOC = \frac{n}{m} = \frac{p}{q}$ bo‘ladi, ya’ni: $qn=mp$ bo‘lib, bunda q va p o‘zaro tub sonlar bo‘lganligi sababli n son p ga, m son esa q ga karrali, ya’ni, $m \geq q, \quad n \geq p$. Ziddiyat. Shuning uchun OBC uchburchakda qaralayotgan butun koordinatali nuqtalarning teng yarmi, ya’ni $\frac{(p-1)(q-1)}{2}$ tasi yotadi.

Endi biz ushbu miqdorni boshqacha usul bilan hisoblaymiz. $x=k$ (k - o‘zgaruvchi natural son) bo‘lsa, u holda KL kesmada jami $\left[\frac{p}{q}k\right]$ ta butun koordinatali nuqta yotadi (3 rasm). $1 \leq x \leq q-1, \quad 1 \leq y \leq p-1$ bo‘lgani uchun k sonni o‘zgartirib, uchburchakda yotgan butun koordinatali nuqtalar umumiy soni quyidagicha aniqlanadi:

$$\left[\frac{p}{q}\right] + \left[\frac{2p}{q}\right] + \ldots + \left[\frac{(q-1)p}{q}\right]$$

Demak,

$$\left[\frac{p}{q}\right] + \left[\frac{2p}{q}\right] + \ldots + \left[\frac{(q-1)p}{q}\right] = \frac{(p-1)(q-1)}{2}$$ tenglik kelib chiqadi [6].

6-masala. Agar $[x] \cdot \{x\} = 100$ ekani ma’lum bo‘lsa, $[x^2] - [x]^2$ ifodaning qiymatini toping.

Yechim. Masalani yuqorida keltirilgan sonning butun va kasr qismi xossalaridan foydalanib yechamiz:

$$[x^2] - [x]^2 = [([x] + \{x\})^2] - [x]^2 = [[x]^2 + 2[x]\{x\} + \{x\}^2] - [x]^2 = [x]^2 + [2[x]\{x\} + \{x\}^2] -$$
$$- [x]^2 = [200 + \{x\}^2] = 200 + [\{x\}^2] = 200$$

Javob: 200. (N.Rahimov, 2023y).

7-masala. $\int_0^2 [x^2]dx$ aniq integralni hisoblang.

Yechim. Bu integralni dastlab oraliqlarga ajratib olamiz:

$$I = \int_0^2 [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^{2} [x^2]dx.$$

$$I = \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{2} 3dx = x\,|_1^{\sqrt{2}} + 2x\,|_{\sqrt{2}}^{\sqrt{3}} + x\,|_{\sqrt{3}}^{2} = (\sqrt{2} - 1) +$$
$$+ 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) = \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} = 5 - \sqrt{3} - \sqrt{2}$$

Javob: $5 - \sqrt{3} - \sqrt{2}$. (N.Raximov, Methods of solving equations related to whole and fractional part of a number., November, 2022y.)

8-masala. Aniq integralni hisoblang: $\int_1^6 [x]dx$

Yechim. $\int_1^6 [x]dx = \int_1^2 dx + \int_2^3 2dx + \int_3^4 3dx + \int_4^5 4dx + \int_5^6 5dx =$

$= x|_1^2 + 2x|_2^3 + 3x|_3^4 + 4x|_4^5 + 5x|_5^6 = 1 + 2 + 3 + 4 + 5 = 15$ [3].

9-masala. Aniq integralni hisoblang: $\int_1^3 \{x^{[x]}\} dx$

Yechim. $\int_1^3 \{x^{[x]}\} dx = \int_1^3 (x - [x])^{[x]} dx = \int_1^2 (x-1)dx + + \int_2^3 (x-2)^2 dx = \left(\frac{x^2}{2} - x\right)\Big|_1^2 + \left(\frac{x^2}{2} - 2x^2 + 4x\right)\Big|_2^3 = 2 - 2 - \frac{1}{2} + 1 + \frac{9}{2} - -18 + 12 - 2 + 8 - 8 = 5$

10-masala. Aniq integralni hisoblang: $\int_1^4 [x]\{x\}dx$

Yechim. $\int_1^4 [x]\{x\}dx = \int_1^4 [x](x - [x])dx = \int_1^2 (x-1)dx + + \int_2^3 2(x-2)dx + \int_3^4 3(x-3)dx = \left(\frac{x^2}{2} - x\right)\Big|_1^2 + 2\left(\frac{x^2}{2} - 2x\right)\Big|_2^3 + +3\left(\frac{x^2}{2} - 3x\right)\Big|_3^4 = \frac{1}{2} - 3 + 4 + 15 - \frac{27}{2} = 3.$ (N.Raximov, O‘quvchilarga sonning butun va kasr qismi qatnashgan tenglamalarni o‘qitish metodikasi. , February 2023.)

11-masala. Aniq integralni hisoblang: $\int_1^4 \frac{\{x\}}{[x]} dx$

Yechim. $\int_1^4 \frac{\{x\}}{[x]} dx = \int_1^4 \frac{x - \{x\}}{[x]} dx = \int_1^2 \frac{x-1}{1} dx + \int_2^3 \frac{x-2}{2} dx + \int_3^4 \frac{x-3}{3} dx = \left(\frac{x^2}{2} - x\right)\Big|_1^2 + \left(\frac{x^2}{4} - x\right)\Big|_2^3 + \left(\frac{x^2}{6} - x\right)\Big|_3^4 = \frac{1}{2} + \frac{1}{4} + \frac{3}{2} - \frac{4}{3} = \frac{11}{12}.$ (N.Raximov, Methods of solving equations related to whole and fractional part of a number., November, 2022y.)

Foydalanilgan adabiyotlar

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